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Solitons and magnons in the classical Heisenberg chain

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Abstract. We review the permanent profile solutions of the continuous classical Heisenberg chain and expound on the application of the inverse scattering method. Extending and amplifying the work of Takhtajan, we exhibit the 'diagonal' action angle representation of the model. The spectrum of the Hamiltonian is exhausted by a magnon band and a soliton band. The magnons have no internal degrees of freedom and can be characterised by the dispersion law $E = p^2$. Like the sine Gordon doublet, the solitons have internal structure, they carry a continuous angular momentum m, and are characterised by the dispersion law $E = 16 \sin^2(p/4)/|m|$, in accordance with Tjon and Wright. The continuous Heisenberg chain is a completely integrable Hamiltonian system possessing an infinite number of constants of motion. We establish the recursive procedure for the determination of the conserved integrated densities.

1. Introduction

In contrast to the quantum Heisenberg chain which, since the pioneering work of Bethe (1931), has been studied extensively (Mattis 1965, Lieb and Mattis 1966), the classical counterpart for a spin or angular momentum field $\bar{S}$ of fixed length has until recently received much less attention. This model, whose dynamics in the long-wavelength limit and in the presence of a magnetic field $\bar{h}$ in convenient units is governed by the precessional equation of motion (Landau and Lifshitz 1935)

$$\frac{\mathrm{d}\bar{S}}{\mathrm{d}t} = \bar{S} \times \frac{\mathrm{d}^2\bar{S}}{\mathrm{d}x^2} + \bar{S} \times \bar{h} \tag{1.1}$$

is, however, of considerable interest, both in its own right as a classical non-linear dynamical system with intriguing properties and in providing an approximative description of magnetic materials exhibiting various kinds of one-dimensional behaviour; in particular, for large values of the quantum spin, i.e. in the semi-classical limit (de Jongh and Miedema 1974, Steiner *et al* 1976). Finally, what prompted the present author to undertake the following investigation is that the model provides the natural starting point for analysing the expected anomalous hydrodynamical behaviour of low-dimensional magnetic systems (Nelson and Fisher 1977).

Special solutions of equation (1.1) have been derived by several authors. Nakamura and Sasada (1974) found analytic expressions for the permanent profile solitary wave and periodic wave train solutions. Lakshmanan *et al* (1976) discussed the spin wave spectrum and derived also the solitary wave solution. Tjon and Wright (1977)

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performed a numerical study and found that a single solitary wave is stable with respect to small perturbations and that two colliding ones preserve their identity, thus providing evidence that the solitary wave is a bona fide soliton (Scott *et al* 1973).

The spin wave is spatially extended, has a uniform energy density, and propagates with a constant amplitude. Its frequency ω and wavenumber k are related by the dispersion law $\omega = S^z k^2 + h$, where S^z is the amplitude and $\bar{h}$ is chosen in the z direction. The solitary wave, on the other hand, is spatially localised, has an energy density peaked at the 'centre of mass' position and a finite total energy E. It carries a linear momentum P, an angular momentum M, and is characterised by the dispersion law $E = 16 \sin^2(P/4)/M$. The wave train has a periodic energy density and can be visualised as a lattice of solitary waves or, alternatively, as a single one in a box with periodic boundary conditions.

The general solution of equation (1.1) for arbitrary initial conditions has been considered by several investigators. Drawing from work by Hasimoto (1972) and Lamb (1976, 1977) on the relationship between the motion of helical curves and non-linear evolution equations, Lakshmanan (1977) has shown that the energy and current densities are given by the solutions of the completely integrable non-linear Schrödinger equation (Zakharov and Shabat 1972). In particular, this relationship enables him to exhibit the stable N-soliton envelope solutions for the energy and current densities from those of the non-linear Schrödinger equation, and to conclude that the continuum spin problem characterised by equation (1.1) is completely integrable and possesses an infinite number of constants of motion. Takhtajan (1977) has shown that equation (1.1) admits a Lax representation and, consequently, falls within the scope of the inverse scattering method, which allows for an exact solution of the arbitrary initial value problem (Scott *et al* 1973, Bishop and Schneider 1979). Takhtajan derives the time dependence of the scattering data for the Lax operator and presents the Gel'fand–Levitan–Marchenko equation for the reconstruction of the spin density, i.e. the inverse scattering problem. Furthermore, he gives the single-soliton solution and the phase and 'centre of mass' shifts induced during two-soliton collisions. He concludes by noting that the Hamiltonian approach developed by Zakharov and Faddeev (1972) for the Korteweg–de Vries equation and by Takhtajan and Faddeev (1975) for the sine Gordon equation allows for the complete integration of the continuous Heisenberg chain in terms of canonical action angle variables, and for the explicit construction of the infinite series of constants of motion. Finally, we mention the work of Corones (1977) who, in a small-amplitude approximation, has shown that the slowly varying non-linear envelope of linear magnons satisfies the non-linear Schrödinger equation, thus, independently, corroborating the conclusions of Lakshmanan (1977).

The purpose of the present paper is two-fold. In the first part, which is of an expository character, we discuss the physical properties of the special permanent profile solutions and, furthermore, expound on the application of the inverse scattering methods to the general initial value problem, following Takhtajan (1977) and hopefully elucidating some of the steps he performs or implies. In the second part we extend and amplify the work of Takhtajan by explicitly carrying out the transformation from the spin variables to the canonical action angle representation, thereby exhibiting the independent magnon and soliton contributions to the transformed Hamiltonian. In constructing the appropriate Poisson bracket relations, characterising the canonical transformation, we follow, in particular, Zakharov and Manakov (1975), who have applied such methods to the non-linear Schrödinger and Korteweg–de Vries equations. We furthermore construct the infinite series of constants of motion using the methods developed by Zakharov and Faddeev (1972) and by Takhtajan and Faddeev (1975) for the Korteweg–deVries and sine Gordon equations, respectively.

The paper is organised in the following manner. In § 2 we introduce the classical isotropic Heisenberg chain and the associated Poisson bracket algebra for the spin field, and derive the continuum representation. In § 3 we embed the continuum model in a Hamiltonian framework and identify the conserved densities associated with the global symmetry transformations. In § 4, which contains three subsections, we discuss the physical properties of the special permanent profile solutions: spin waves (§ 4.1), solitary waves (§ 4.2) and periodic wave trains (§ 4.3). In § 5, which includes four subsections, we carry out a general dynamical analysis by means of the inverse scattering method. We discuss the Lax representation (§ 5.1) and the associated eigenvalue problem of the Lax operator (§ 5.2). We deduce the time dependence of the scattering data for the Lax operator (§ 5.3) and, finally, derive the Gel'fand–Levitan–Marchenko equation for the reconstruction of the spin field from the time-dependent scattering data (§ 5.4). In § 6 we construct, using the inverse scattering method, the canonical transformation to the action angle representation. In order to identify the dynamical modes we represent in § 7 the total energy, the total momentum and the total angular momentum in the action angle representation, and discuss the spectrum of solitons and magnons in some detail. In § 8 we construct the infinite series of conserved densities. We end in § 9 with a summary and a conclusion. Since a proper discussion of the model requires a fair amount of analysis, we have deferred encumbering technical and mathematical aspects to six appendices.

Below we summarise our main result. Assuming that the ground state of the continuous Heisenberg chain has the constant spin field pointing in the z direction and, furthermore, imposing the fixed boundary condition $S^z \to 1$ for $|x| \to \infty$ and choosing a rotating frame such that $\bar{h} = 0$, the Hamiltonian H, the total momentum P, and the z component of the total angular momentum M^z, i.e. the three constants of motion associated with the global symmetries of the time translation, space translation and spin rotation, respectively, are in the action angle representation given by the 'diagonal' expressions

$$H = \int d\lambda \, n(\lambda)\omega(\lambda) + \sum_{n=1}^{M} E_n \tag{1.2a}$$

$$P = \int d\lambda \, n(\lambda)p(\lambda) + \sum_{n=1}^{M} P_n \tag{1.2b}$$

$$M^z = -\int d\lambda \, n(\lambda) + \sum_{n=1}^{M} m_n. \tag{1.2c}$$

Here $n(\lambda)$, determined by the initial conditions, is the density of magnon modes in λ space. In units of $n(\lambda)$ the magnons have a quadratic dispersion law $\omega(\lambda) = \pi(\lambda)^2$, $\omega(\lambda) = 4\lambda^2$ and $p(\lambda) = 2\lambda$, and carry unit angular momentum. The energy E_n, the momentum p_n and the angular momentum m_n of the discrete soliton modes, likewise specified by the initial conditions for the spin field, are related by the dispersion law $E_n = 16 \sin^2(p_n/4)/|m_n|$, i.e. the same as for the solitary waves. Like the sine Gordon 'breather' (Bishop and Schneider 1979), the solitons have internal structure; they carry an angular momentum. For small momentum the 'magnetic' solitons have an effective mass proportional to their angular momentum.

2. The model

The classical ferromagnetic Heisenberg chain in a constant field is governed by the Hamiltonian

$$H = -J\sum_n \bar{S}_n\bar{S}_{n+1} - \bar{H}_0\sum_n \bar{S}_n \tag{2.1}$$

where J is a positive nearest-neighbour exchange coupling and H_0^α, $\alpha = x, y, z$, is a magnetic field, both of dimension energy. The dimensionless spin field S_n^α, $\alpha = x, y, z$, measured in units of action, associated with the site n of a one-dimensional lattice, $n = 1, \ldots, N$, with lattice parameter a, is assumed scaled to unit length, that is $\Sigma_\alpha S_n^{\alpha^2} = 1$.

The Hamiltonian H is the generator of time translations (Landau and Lifshitz 1960). For the spin field, in particular, we thus obtain the equation of motion

$$\mathrm{d}\bar{S}_n/\mathrm{d}t = \{H, \bar{S}_n\} \tag{2.2}$$

where $\{A, B\}$ denotes the Poisson bracket (Landau and Lifshitz 1960)

$$\{A, B\} = \sum_{n\alpha}\left(\frac{\mathrm{d}A}{\mathrm{d}p_n^\alpha}\frac{\mathrm{d}B}{\mathrm{d}q_n^\alpha} - \frac{\mathrm{d}B}{\mathrm{d}p_n^\alpha}\frac{\mathrm{d}A}{\mathrm{d}q_n^\alpha}\right), \tag{2.3}$$

defined in terms of an underlying canonical basis (q_n^α, p_n^α), $\alpha = x, y, z$, pertaining to each site of the lattice. Since the composite spin field has the dynamical structure of an angular momentum, that is $\bar{S}_n = \bar{q}_n \times \bar{p}_n$, we infer the non-canonical Poisson bracket relations (Landau and Lifshitz 1960)

$$\{S_n^\alpha, S_m^\beta\} = -\delta_{nm}\sum_\gamma \epsilon^{\alpha\beta\gamma} S_n^\gamma \tag{2.4}$$

and, as a result, inserting equation (2.1) in equation (2.2), the equation of motion

$$\mathrm{d}\bar{S}_n/\mathrm{d}t = J\bar{S}_n \times (\bar{S}_{n+1} + \bar{S}_{n-1}) + \bar{S}_n \times \bar{H}_0. \tag{2.5}$$

Since the non-linear difference character of equation (2.5) renders a general discussion difficult, it is expedient, for the purpose of investigating the properties of the Heisenberg chain at wavelengths much larger than the lattice distance, to replace the Hamiltonian (2.1) by the continuum form

$$H = \tfrac{1}{2}\int \mathrm{d}x \left(\frac{\mathrm{d}\bar{S}}{\mathrm{d}x}\right)^2 - h\int \mathrm{d}x\, (S^z - 1) \tag{2.6}$$

obtained by assuming a slow variation of $\bar{S}_n$ over a lattice distance and expanding $\bar{S}_n \simeq \bar{S}(x)$, keeping only leading-order terms. We have, furthermore, measured lengths in units of the lattice parameter a, energies in units of the exchange constant J, chosen the dimensionless field $\bar{h} = \bar{H}_0/J$ in the positive z direction, and subtracted the ground-state energy. The continuum spin field $\bar{S}(x)$ now satisfies the Poisson bracket algebra

$$\{S^\alpha(x), S^\beta(y)\} = -\delta(x-y)\sum_\gamma \epsilon^{\alpha\beta\gamma} S^\gamma(x) \tag{2.7}$$

and equation (2.5), correspondingly, takes the form

$$\frac{\mathrm{d}\bar{S}}{\mathrm{d}t} = \bar{S} \times \frac{\mathrm{d}^2\bar{S}}{\mathrm{d}x^2} + \bar{S} \times \bar{h} \qquad \bar{h} = (0, 0, h). \tag{2.8}$$

The precessional equation of motion (2.8) was first derived on phenomenological grounds by Landau and Lifshitz (1935) and later, independently, by Döring (1947). The approach by means of Poisson brackets is due to Mermin (1964, 1967). We stress that the continuum form (2.6) only correctly samples the long-wavelength spin configurations of the Heisenberg chain (2.1). In the continuum limit 'neighbouring' spins deviate only little with respect to one another. The spin field, however, 'floats' over all directions, as indicated in figure 1, where we have shown an arbitrary spin configuration.

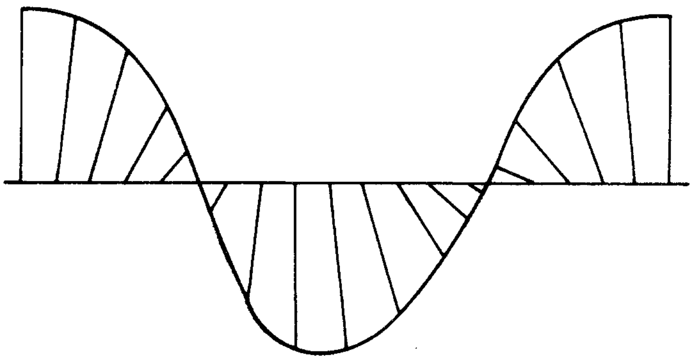

Figure 1. Arbitrary spin configuration with envelope shown.

3. Hamiltonian formulation—constants of motion

In order to exhibit the non-linear character of the spin problem, caused by the precessional self-coupling $\bar{S} \times \mathrm{d}^2\bar{S}/\mathrm{d}x^2$, within the framework of Hamiltonian dynamics (Landau and Lifshitz 1960), we introduce canonical variables. This also enables us to identify easily the constants of motion associated with the global symmetries of the Hamiltonian (2.6).

Following Tjon and Wright (1977) we choose as canonical coordinate $q(x)$ the local azimuthal angle $\phi(x)$ in a polar coordinate basis $(\phi(x), \theta(x))$ of the spin field $\bar{S}(x)$. In analogy with the quantum treatment of a spin (Landau and Lifshitz 1958), the corresponding canonical momentum $p(x)$ is then given by the z component of the spin field, $S^z(x) = \cos\theta(x)$. Introducing $S^{\pm} = S^x \pm \mathrm{i}S^y$,

$$S^+(x) = (1-p(x)^2)^{1/2}\exp(\mathrm{i}q(x)) \qquad S^z(x) = p(x) \tag{3.1}$$

and it follows from equation (2.7) that $p(x)$ and $q(x)$ satisfy the canonical Poisson brackets

$$\{p(x), q(y)\} = \delta(x-y) \qquad \{p(x), p(y)\} = \{q(x), q(y)\} = 0. \tag{3.2}$$

In terms of the variables $p(x)$ and $q(x)$ the Hamiltonian (2.6) takes the form

$$H = \tfrac{1}{2}\int \mathrm{d}x\left[\frac{1}{1-p^2}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)^2 + (1-p^2)\left(\frac{\mathrm{d}q}{\mathrm{d}x}\right)^2\right] - h\int \mathrm{d}x(1-p). \tag{3.3}$$

The corresponding equations of motion, which, of course, are equivalent to the field equation (2.8), are given by

$$\frac{\mathrm{d}q}{\mathrm{d}t}=\{H,q\}=-\frac{1}{1-p^2}\frac{\mathrm{d}^2p}{\mathrm{d}x^2}-\frac{p}{(1-p^2)^2}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)^2-p\left(\frac{\mathrm{d}q}{\mathrm{d}x}\right)^2-h \tag{3.4a}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t}=\{H,p\}=(1-p^2)\frac{\mathrm{d}^2q}{\mathrm{d}x^2}-2p\frac{\mathrm{d}p}{\mathrm{d}x}\frac{\mathrm{d}q}{\mathrm{d}x}. \tag{3.4b}$$

Unlike the Hamiltonian for a particle system, $H=T(p)+U(q)$, which is a sum of a kinetic energy $T(p)$ and a potential energy $U(q)$, the form of equation (3.3) does not allow for a simple particle-like interpretation and shows the strong intrinsic non-linear character of the spin problem.

The total momentum Π is the generator of space translations and is defined by the Poisson bracket $\mathrm{d}F/\mathrm{d}x=-\{\Pi,F\}$ (Landau ancd Lifshitz 1960), where F is an arbitrary function of p and q. For the spin density, in particular, we have

$$\mathrm{d}\bar{S}/\mathrm{d}x=-\{\Pi,\bar{S}\}. \tag{3.5}$$

The Poisson bracket of Π with H for a 'string' of length $2L$, $\{\Pi,\mathrm{H}\}=[\frac{1}{2}(\mathrm{d}\bar{S}/\mathrm{d}x)^2-hS^z]_{x=-L}^{x=L}$, vanishes provided we impose either fixed boundary conditions $S^z\to 1$ and $\mathrm{d}\bar{S}/\mathrm{d}x\to 0$ for $|x|=L\to\infty$ or periodic boundary conditions $S^z(L)=S^z(-L)$ and $(\mathrm{d}\bar{S}/\mathrm{d}x)_{x=L}=(\mathrm{d}\bar{S}/\mathrm{d}x)_{x=-L}$, in which case the Hamiltonian is translationally invariant and, since $\{\Pi,H\}=0$, the total momentum is a constant of motion. In the basis (3.1*a*–*c*) Π is given by the integrated density (Tjon and Wright 1977)

$$\Pi=\int \mathrm{d}x(1-p)\,\mathrm{d}q/\mathrm{d}x \tag{3.6}$$

where we, in order to ensure a vanishing Π in the ground state $S^z=p=1$, have subtracted a total derivative.

The total spin or angular momentum $\bar{M}$ is the generator of rotations in spin space. It is defined by the Poisson bracket $\{M^\alpha,\ S^\beta(x)\}=-\Sigma_\gamma\epsilon^{\alpha\beta\gamma}S^\gamma(x)$ (Landau and Lifshitz 1960), and it follows from equation (2.7) that

$$M^x=\int \mathrm{d}x\,S^x(x)\qquad M^y=\int \mathrm{d}x\,S^y(x)\qquad M^z=\int \mathrm{d}x(S^z(x)-1) \tag{3.7}$$

where we have subtracted the ground-state value. In the basis (3.1*a*–*c*) $\bar{M}$ takes the form

$$M^+=\int \mathrm{d}x(1-p^2)^{1/2}\exp(\mathrm{i}q)\qquad M^z=\int \mathrm{d}x(p-1). \tag{3.8}$$

For $\bar{h}=0$ the Hamiltonian (2.6) is invariant under rotations in spin space. Consequently, the components of $\bar{M}$ are constants of motion, i.e. $\{\bar{M},H\}=0$. In the presence of the field the Hamiltonian remains invariant under rotations about the z axis. Hence, $\{M^z,H\}=0$ and only M^z is a constant of motion.

4. Permanent profile solutions

Prior to discussing the general dynamical solution of the continuous Heisenberg chain (2.6), it is instructive to consider a class of special solutions which can be derived by quadrature, namely spin configurations propagating with a permanent profile.

We search, in other words, for solutions of the form $\bar{S}(xt)=\bar{S}(x-vt)$, where v is the phase velocity of the permanent profile. In terms of the equations of motion (3.4*a*–*b*) we obtain, inserting $q=q(x-vt)$ and $p=p(x-vt)$,

$$-v\frac{\mathrm{d}q}{\mathrm{d}x}=-\frac{1}{1-p^2}\frac{\mathrm{d}^2p}{\mathrm{d}x^2}-\frac{p}{(1-p^2)^2}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)^2-p\left(\frac{\mathrm{d}q}{\mathrm{d}x}\right)^2-h$$

$$-v\frac{\mathrm{d}p}{\mathrm{d}x}=(1-p^2)\frac{\mathrm{d}^2q}{\mathrm{d}x^2}-2p\frac{\mathrm{d}p}{\mathrm{d}x}\frac{\mathrm{d}q}{\mathrm{d}x}.$$

The second equation is readily integrated once:

$$\frac{\mathrm{d}q}{\mathrm{d}x}=v\frac{p_0-p}{1-p^2}. \tag{4.1}$$

Substituting equation (4.1) we arrive at a second-order equation for p which, integrated once, yields

$$(\mathrm{d}p/\mathrm{d}x)^2=F(p)=2hp(p^2-1)-v^2(1+p_0-2p_0p)-p_1(p^2-1) \tag{4.2}$$

where p_0 and p_1 are constants of integration. The general solution of equation (4.2) is given by an elliptic function (Whittaker and Watson 1962). For the present purposes, however, we limit ourselves to an elementary discussion.

In order to obtain a solution of equation (4.2) we must choose p_0 and p_1 such that the cubic polynomial $F(p)$ is positive for p in the admissible range $|p|<1$. Depending on the positions of the roots of $F(p)$ and its sign, we can distinguish three main cases.

(i) $F(p)$ is negative except for a double root p_A. In this case p is tied to the value p_A and equation (4.1) has the solution $q=k(x-vt)+q_0$, where $k=(p_0-p_A)\,(1-p_A^2)^{-1}$ and q_0 is an integration constant. In spin space $S^z=p_A$, $S^+=(1-p_A^2)^{1/2}\exp(\mathrm{i}q)$ the above solution corresponds to the propagation of a spin wave with wavenumber k, frequency vk and a constant amplitude p_A.

(ii) $F(p)$ has a single root p_A and a double root at one of the end points, say $p=1$, is positive for $p_A<p<1$ and otherwise negative. In this case equation (4.2) has a turning point (Jeffreys and Jeffreys 1972) at $p=p_A$ and a degenerate one at $p=1$. The motion of p is restricted to the interval $p_A<p<1$. For $|x|\to\infty$ p approaches the ground-state value $+1$. In order to ensure a solution of equation (4.1) we choose $p_0=1$, i.e. $q=v\int^{x-vt}\mathrm{d}y\,(1+p(y))^{-1}$, and for $\mathrm{d}p/\mathrm{d}x$ to vanish for $|x|\to\infty$ $p_1=v^2+2h$. In spin space this solution corresponds to the propagation of a solitary wave with phase velocity v and amplitude $1-p_A$.

(iii) $F(p)$ has single roots at p_A and p_B, is positive for $p_A<p<p_B$, and otherwise negative. In this case equation (4.2) has turning points at both p_A and p_B. The motion of p is confined to the interval $p_A<p<p_B$ and is oscillatory with period $2\int_{p_A}^{p_B}\mathrm{d}p\,F(p)^{-2}$. The same applies to the solution of equation (4.1), $q=v\int^{x-vt}\mathrm{d}y\,(p_0-p(y))\,(1-p(y)^2)^{-1}$. In spin space the above solution corresponds to the propagation of a periodic wave train with phase velocity v and amplitude p_B-p_A. As the period diverges, the wave train reduces to a single solitary wave.

Finally, we remark that the case where equation (4.2) has a turning point at p_A and a degenerate one at p_B is a variant of case (ii), corresponding to a rotation of the spin frame, i.e. a solitary wave with p approaching the ground-state value p_B for $|x|\to\infty$. Also, for certain values of p_0 and p_1 the periodic wave train in case (iii) degenerates to a spin wave in a rotated frame.

4.1. *Spin waves*

As discussed above, the spin wave has a constant amplitude. From the equation of motion (2.8) we thus obtain $\mathrm{d}S^+/\mathrm{d}t = \mathrm{i}S^z\,\mathrm{d}^2S^+/\mathrm{d}x^2 - \mathrm{i}S^+h$, subject to the constraint $S^-\,\mathrm{d}^2S^+/\mathrm{d}x^2 = S^+\,\mathrm{d}^2S^-/\mathrm{d}x^2$. Imposing, furthermore, the length condition $S^+S^- + (S^z)^2 = 1$ we infer, in accordance with Lakshmanan *et al* (1976), the spin wave solution

$$S^+(xt) = (1-(S^z)^2)^{1/2}\exp[\mathrm{i}(kx-\omega t+\phi)] \tag{4.3}$$

specified by the amplitude S^z, the wavenumber k, the field h and the phase ϕ. The frequency ω is given by the dispersion law:

$$\omega = S^zk^2 + h \tag{4.4}$$

which depends quadratically on k and has a gap h. Unlike a quantum spin wave (Bloch 1930), which is an elementary excitation, the classical counterpart forms a band, even for a fixed value of k, as shown in figure 2, where we have plotted ω versus k for $|S^z| \leq 1$. We also remark that for $h = -S^zk^2$, i.e. $|h| \leq k^2$, we have a band of static spin waves.

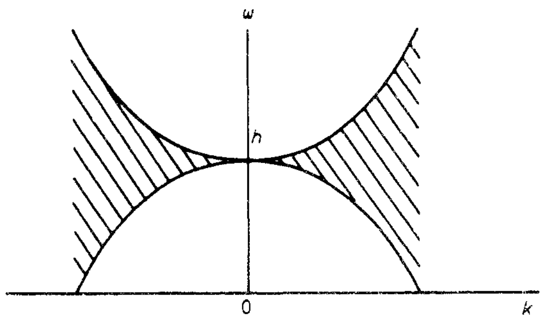

Figure 2. Plot of $\omega = S^zk^2 + h$. The shaded area indicates the spin wave band (arbitrary units).

The energy density of the spin wave is obtained by substituting equation (4.3) in equation (2.6):

$$\epsilon = \tfrac{1}{2}k^2[1-(S^z)^2] - h(S^z-1) \tag{4.5}$$

i.e. the total energy of the non-localised wave is infinite. We note that the largest energy density is attained by the band of static spin waves for $h = -S^zk^2$.

We stress that the spin wave spectrum discussed here is an exact solution of the equation of motion (2.8). By considering small deviations from the aligned ground state $S^z = 1$ we obtain a linearised spectrum with dispersion law $\omega = k^2 + h$. It is interesting to notice that the sole effect of the local mode coupling $\tilde{S}\times\mathrm{d}^2\tilde{S}/\mathrm{d}x^2$, which in wavenumber space gives rise to harmonic generations, is to change the stiffness coefficient from unity to S^z.

4.2. *Solitary waves*

In order to derive the solitary wave solution (Nakamura and Sasada 1974, Lakshmanan *et al* 1976, Tjon and Wright 1977) we choose, as discussed above, $p_0 = 1$ and $p_1 = v^2 + 2h$ in the equations of motion (4.1) and (4.2), i.e. $\mathrm{d}q/\mathrm{d}x = v(1+p)^{-1}$ and

$(\mathrm{d}p/\mathrm{d}x)^2 = (p-1)^2[2h(p+1) - v^2]$. In polar coordinates $p = \cos\theta$ and $q = \phi$ we have $\mathrm{d}\phi/\mathrm{d}x = v(1+\cos\theta)^{-1}$ and $(\mathrm{d}\theta/\mathrm{d}x)^2 = 2h(1-\cos\theta) - v^2(1+\cos\theta)(1-\cos\theta)^{-1}$ which, introducing the half-angle $\theta/2$, are readily solved by quadrature (Tjon and Wright 1977). We thus obtain the solitary wave solution

$$\cos\theta(xt) = 1 - \frac{A}{\cosh^2[(x-vt-x_0)/\Gamma]} \tag{4.6a}$$

$$\phi(xt) = \phi_0 + \tfrac{1}{2}v(x-vt-x_0) + \tan^{-1}\left(\frac{2}{v\Gamma}\tanh[(x-vt-x_0)/\Gamma]\right) \tag{4.6b}$$

where we have introduced the amplitude A and the width Γ:

$$A = 2 - \frac{v^2}{2h} \qquad \Gamma = \frac{1}{[h-(v/2)^2]^{1/2}}. \tag{4.7}$$

The 'centre of mass' x_0 and the phase ϕ_0 are determined by the initial conditions.

The solitary wave mode (4.6*a*–*b*) is conveniently characterised by the four parameters A, Γ, x_0 and ϕ_0. Unlike the spin wave velocity ω/k which has an unlimited range, the phase velocity of the solitary wave is restricted to the interval $v^2 \leq 4h$. In the low-velocity limit the amplitude attains its maximum value 2, and the width its smallest value $1/h^{1/2}$. On the other hand, as the velocity approaches the limiting values $\pm 2h^{1/2}$, the amplitude vanishes while the width diverges, i.e. the solitary wave disappears. This behaviour is different from both the Korteweg–deVries and the sine Gordon solitary waves, which become sharper, and in the Korteweg–deVries case larger, as the velocity increases (Scott *et al* 1973). In figures 3 and 4 we have depicted the longitudinal component of the solitary wave for a large and a small velocity, respectively.

The phase or azimuthal angle ϕ, given by equation (4.6*b*), is essentially a linear function of $x - vt$ at the leading and trailing edges of the solitary wave. However, as we

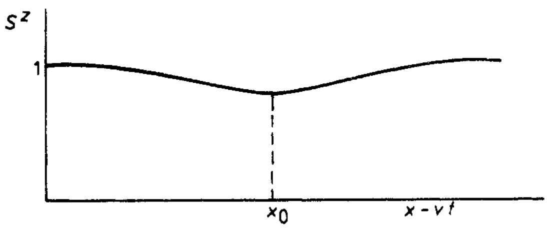

Figure 3. Longitudinal component of small-amplitude–large-width–large-velocity solitary wave (arbitrary units).

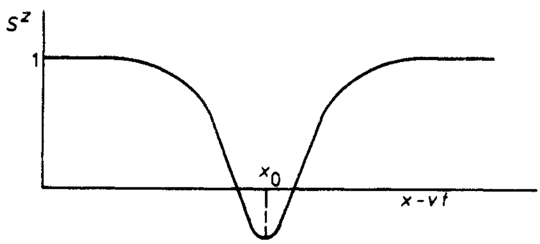

Figure 4. Longitudinal component of large-amplitude–small-width–small-velocity solitary wave (arbitrary units).

move across a region of width Γ about the centre of the wave, the angle ϕ is augmented by a positive phase shift $\Delta\phi$,

$$\Delta\phi = 2\tan^{-1}(2/v\Gamma) \tag{4.8}$$

as shown in figure 5. As the velocity approaches zero the phase shift attains its maximum value π; for $v^2 = 4h$ the phase shift vanishes together with the solitary wave. In figure 6 we have plotted $\Delta\phi$ versus v. Furthermore, in order to illustrate the phase shift effect we have in figures 7–9 depicted the transverse component and the corresponding envelope in the three cases of a small amplitude, a half amplitude, and a full amplitude solitary wave.

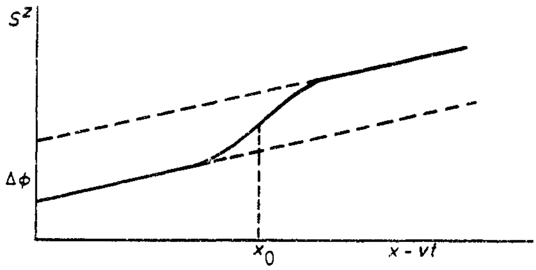

Figure 5. The phase ϕ versus $x - vt$ showing the phase shift $\Delta\phi$ (arbitrary units).

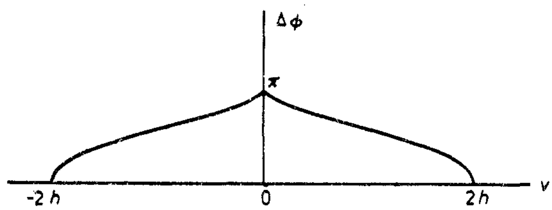

Figure 6. The phase shift $\Delta\phi$ versus the phase velocity v (arbitrary units).

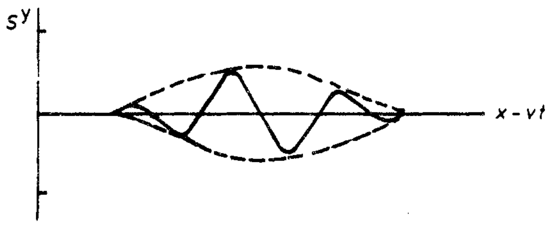

Figure 7. Transverse component of small-amplitude solitary wave with envelope shown (arbitrary units).

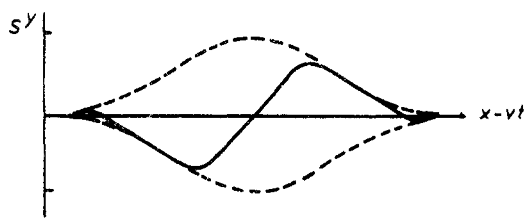

Figure 8. Transverse component of half-amplitude solitary wave with envelope shown (arbitrary units).

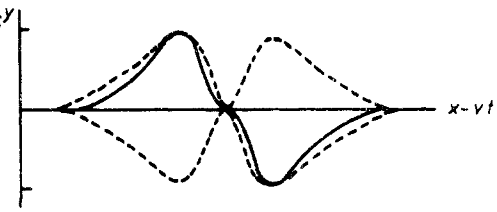

Figure 9. Transverse component of full-amplitude solitary wave with envelope shown (arbitrary units).

The energy density of a solitary wave, obtained by substituting equation (4.6*a–b*) in equation (3.3),

$$\epsilon(xt) = \frac{4/\Gamma^2}{\cosh^2[(x - vt - x_0)/\Gamma]} \tag{4.9}$$

is, unlike the spin wave case, peaked at the 'centre of mass' position. The finite total energy

$$E = \frac{8}{\Gamma} = 8\left[h - \left(\frac{v}{2}\right)^2\right]^{1/2} \tag{4.10}$$

is inversely proportional to the width. In figure 10 we have plotted the energy density.

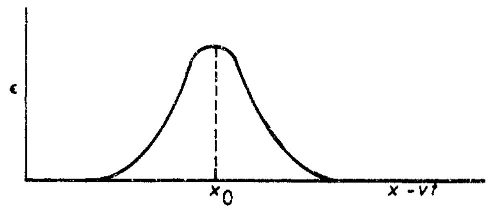

Figure 10. Energy density of solitary wave (arbitrary units).

The total momentum of a solitary wave, evaluated by inserting equation (4.6*a–b*) in equation (3.6),

$$\Pi = 4\sin^{-1}[(A/2)^{1/2}] = 4\sin^{-1}[(1 - v^2/4h)^{1/2}] \tag{4.11}$$

is restricted to the interval $|\Pi| \leq 2\pi$. This peculiar feature is presumably related to the inherent length scale in the continuum model (2.6); introducing the lattice parameter a, the critical momentum is given by $2\pi/a$. The total momentum assumes its maximum value 2π for a full-amplitude wave, the value π for a half-amplitude wave, and vanishes together with the solitary wave. In contrast to the Korteweg–deVries and sine Gordon solitary waves (Bishop and Schneider 1979) the momentum velocity relationship (4.11) or, equivalently, $|v| = 2h^{1/2}\cos(\Pi/4)$, is unusual in that v attains its maximum value $2h^{1/2}$ for vanishing Π, and vice versa. This peculiar relationship is shown in figure 11. Finally, the z component of the total angular momentum of a solitary wave, derived by substituting equations (4.6*a–b*) in equation (3.8),

$$M^z = -\frac{4}{h\Gamma} = -4\frac{[h - (v/2)^2]^{1/2}}{h} \tag{4.12}$$

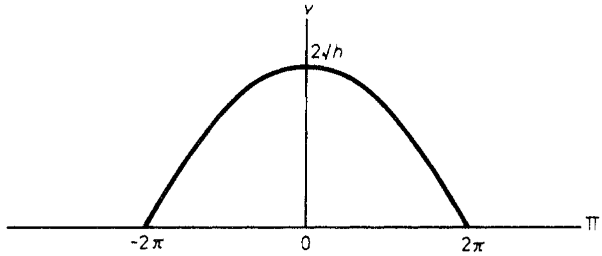

Figure 11. Velocity–momentum relationship for a solitary wave (arbitrary units).

is inversely proportional to the width and the magnetic field, and attains its largest numerical value $4/h^{1/2}$ for $v=0$.

Instead of characterising the solitary wave by the amplitude and the width, we can use the above constants of motion. In accordance with Tjon and Wright (1977), E, Π and M^z satisfy the dispersion law

$$E=\frac{16}{|M^z|}\sin^2\left(\frac{\pi}{4}\right)+h|M^z|=\frac{32}{|M^z|}\sin^2\left(\frac{\pi}{4}\right) \tag{4.13}$$

obtained from equations (4.10)–(4.12) by elimination of A and Γ, using $h=2/A\Gamma^2$. The solitary wave has a continuous internal degree of freedom; it carries an angular momentum. In the low momentum limit $\Pi \ll 1$, $E=\Pi^2/|M^z|$, i.e. the solitary wave propagates as a free particle with effective mass $|M^z|/2$. This analogy ceases, however, to be valid for larger values of Π, in particular, as we approach the critical momentum 2π. In figure 12 we have plotted the dispersion law (4.13) for different values of $|M^z|$. We note, however, that, similar to the spin wave spectrum depicted in figure 2, the solitary waves form a band since $|M^z|$ has a continuous range, as also indicated in figure 12.

Finally, we remark that in the small-amplitude–large-width limit, i.e. from equation (4.7) $v^2 \to 4h$, the solitary wave approaches the spin wave mode for $S^z \to 1$. From equations (4.3) and (4.6b) we obtain $\omega=2h$ and $k=h^{1/2}$, in agreement with the

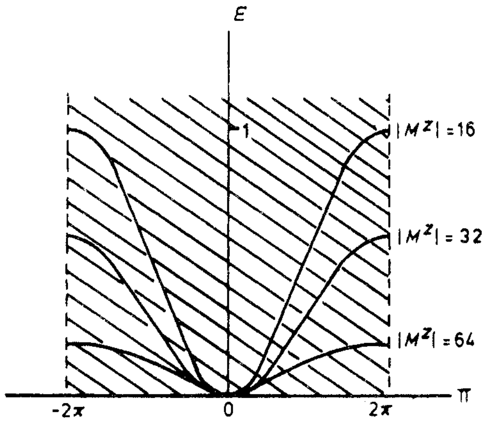

Figure 12. Plot of $E=16\sin^2(\Pi/4)/|M^z|$ for $|M^z|=16$, 32 and 64. The shaded area indicates the solitary wave band.

dispersion law (4.4) for $S^z \to 1$. Notice, however, that the spin wave frequency is determined by the magnetic field. When deforming an arbitrary spin wave, one obtains in general the periodic wave train solution described in the next paragraph. It is indeed instructive (Corones 1977) to regard the solitary wave solution as an amplitude- and phase-modulated spin wave with a single node.

4.3. Wave train

The wave train solution is given by an elliptic function (Nakamura and Sasada 1974), and we shall not discuss the analytic structure here. From the above discussion of the solitary wave solution we can, however, draw some simple conclusions. The wave train can be visualised as a periodic lattice of identical co-moving solitary waves with an energy density peaked at the 'centre of mass' positions. The phase ϕ increases essentially linearly with $x - vt$ in the regions between the peaks; across a peak ϕ is augmented by a phase shift $\Delta\phi$, approximately given by equation (4.9). In the limit where the period of the wave train becomes large the ground state is nearly completely established in the regions between the peaks. In figure 13 we have depicted the transverse component and the corresponding envelope of a periodic wave train. We also remark that the wave train with period L is essentially equivalent to subjecting a solitary wave to periodic boundary conditions in a box of size L, as also indicated in figure 13.

We finally notice that, regarded as an amplitude- and phase-modulated spin wave, the wave train, of course, corresponds to the case of infinitely many periodically spaced nodes, the period being related to the wavenumber of the spin wave.

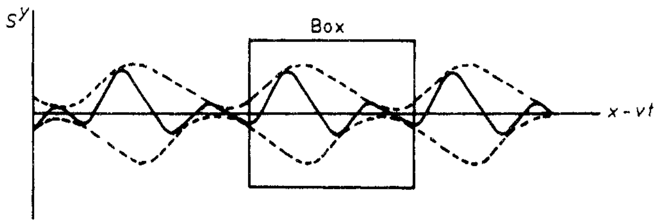

Figure 13. Transverse component of periodic wave train with envelope and box shown (arbitrary units).

5. General dynamical analysis (inverse scattering method)

For the majority of non-linear dynamical systems in one or higher dimensions an exact solution of the arbitrary initial value problem is in general inaccessible. However, as shown by Lakshmanan (1977) and Takhtajan (1977), the continuous Heisenberg chain belongs to the interesting class of non-linear one-dimensional systems which are completely integrable by means of the inverse scattering method (Scott *et al* 1973, Bishop and Schneider 1979).

5.1. The Lax representation

The essential step in applying the inverse scattering method is the introduction of the Lax representation. Following Takhtajan (1977) we embed the equation of motion

(2.8) in the Pauli matrix basis σ^α, $\sigma^\alpha\sigma^\beta = \delta^{\alpha\beta} + \mathrm{i}\,\Sigma_\gamma\,\epsilon^{\alpha\beta\gamma}\sigma^\gamma$ (Landau and Lifshitz 1958). Introducing the spin matrix $S = \Sigma_\alpha\,\sigma^\alpha S^\alpha$, equation (2.8) assumes the form

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\frac{\mathrm{i}}{2}\left[S, \frac{\mathrm{d}^2S}{\mathrm{d}x^2}\right] \qquad S = \begin{Bmatrix} S^z & S^- \\ S^+ & -S^z \end{Bmatrix} \tag{5.1}$$

where we, without loss of generality, have set $h = 0$ since the magnetic field term in equation (2.8) can be absorbed by a transformation to a rotating frame, $S^+ \to S^+\exp(-\mathrm{i}ht)$, i.e. by adding a constant frequency term ht to the azimuthal angle.

In order to 'monitor' the instantaneous spin configuration we consider the associated eigenvalue problem $\mathrm{i}L\Psi = \lambda\Psi$. The operator $L = S\mathrm{d}/\mathrm{d}x$, acting in a space of x dependent matrices Ψ, is a function of the spin field and is therefore time-dependent. However, as has been shown by Takhtajan, the equation of motion (5.1) implies that the spectrum $\{\lambda\}$ of L is independent of time; the eigenfunction Ψ, on the other hand, evolves in time according to $\mathrm{i}\mathrm{d}\Psi/\mathrm{d}t = M\Psi$, where $M = 2S\mathrm{d}^2/\mathrm{d}x^2 + (\mathrm{d}S/\mathrm{d}x)\mathrm{d}/\mathrm{d}x$. For a proof of the above assertions we refer to appendix 1; see also Scott *et al* (1973).

The importance of the Lax representation (L, M) lies in the fact that it replaces the in general intractable problem of solving the non-linear equation of motion directly, by the solution of two linear operator problems

$$\mathrm{i}S\frac{\mathrm{d}\Psi}{\mathrm{d}x} = \lambda\Psi \qquad \mathrm{i}\frac{\mathrm{d}\Psi}{\mathrm{d}t} = 2S\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + \frac{\mathrm{d}S}{\mathrm{d}x}\frac{\mathrm{d}\Psi}{\mathrm{d}x} \tag{5.2}$$

where the spectrum $\{\lambda\}$ is a constant of motion, and the subsequent reconstruction of the spin field. Since there exists a variety of mathematical techniques for linear problems, this represents a major simplification. A representation of the kind (5.2) was first introduced for the Korteweg–deVries equation in the pioneering work by Gardner *et al* (1967), and later refined by Lax (1968), hence the name. Since that time Lax representations have been found for a whole class of one-dimensional non-linear evolution equations (McLaughlin 1975), comprising among others the non-linear Schrödinger equation (Zakharov and Shabat 1972) and the sine Gordon equation (Ablowitz *et al* 1973).

5.2. The eigenvalue problem of the Lax operator

By means of the identity $S^2 = I$, which follows from the length condition $\Sigma_\alpha\,S^{\alpha^2} = 1$, we express the eigenvalue problem of the Lax operator L in the form

$$\mathrm{i}\frac{\mathrm{d}\Psi}{\mathrm{d}x} = \lambda S\Psi. \tag{5.3}$$

In analogy with the spectral theory of the one-dimensional Schrödinger equation (Landau and Lifshitz 1958, Faddeev 1963) which, incidentally, is the associated Lax operator problem for the Korteweg–deVries equation (Zakharov and Faddeev 1972), we distinguish, in the case of fixed boundary conditions at infinity for the spin field, that is $S(x) \to \sigma^z$ for $|x| \to \infty$, equivalent to the infinite-volume limit of the continuum chain, two kinds of solutions to equation (5.3): scattering solutions corresponding to a band of real eigenvalues $-\infty < \lambda < \infty$ $\mathrm{Im}\lambda = 0$, and bound-state solutions which, since L is non-Hermitian, are characterised by discrete complex eigenvalues λ_n, $n = 1, \ldots, M$. The spin field $S(x)$ acts as a 'potential' giving rise to 'wavefunctions' $\Psi(x)$ with different asymptotic behaviour. In the important case of periodic boundary conditions, i.e.

$S(x)=S(x+L)$, corresponding to enclosing the continuum chain in a box of size L, Bloch's theorem implies that the spectrum consists of 'allowed' and 'forbidden' bands. As indicated by the work of Dubrovin and Novikov (1975) on the Korteweg–deVries equation, a discussion of equation (5.3) in the periodic case is, however, technically difficult and will not be considered here.

Referring to appendix 2 for mathematical details, we give below a brief discussion of the eigenvalue problem. Following Takhtajan, we introduce two special solutions of equation (5.3), the Jost functions $F(x\lambda)$ and $G(x\lambda)$ (Jost 1947) determined by boundary conditions at infinity, i.e.

$$F(x\lambda)\to\exp(-\mathrm{i}\lambda\sigma^z x) \qquad \text{for } x\to+\infty \tag{5.4a}$$

$$G(x\lambda)\to\exp(-\mathrm{i}\lambda\sigma^z x) \qquad \text{for } x\to-\infty. \tag{5.4b}$$

The columns (F_{12}, F_{22}) and (G_{11}, G_{21}) are analytic in the upper-half complex λ plane, whereas the columns (F_{11}, F_{21}) and (G_{12}, G_{22}) are analytic in the lower-half plane. Moreover, assuming that $S(x)$ approaches the ground-state matrix σ^z faster than $\exp(-4\lambda_0|x|)$ for $|x|\to\infty$, both F and G are analytic in the Bargmann strip $|\mathrm{Im}\lambda|<\lambda_0$ (Bargmann 1949). Since equation (5.3) is of first order F and G are related by a constant matrix, i.e.

$$G(x\lambda)=F(x\lambda)T(\lambda) \tag{5.5}$$

where the transition matrix $T(\lambda)$, characterising the eigenvalue problem, has the form

$$T(\lambda)=\begin{Bmatrix} a(\lambda) & -(b(\lambda^*))^* \\ b(\lambda & (a(\lambda^*))^* \end{Bmatrix} \qquad a(\lambda)(a(\lambda^*))^*+b(\lambda)(b(\lambda^*))^*=1. \tag{5.6}$$

From the analytic properties of F and G it then follows that $a(\lambda)$ is analytic in the upper-half plane, including the Bargmann strip, and that $b(\lambda)$ is analytic at least in the Bargmann strip.

The scattering solutions for real λ are characterised by the transmission and reflection of an incoming wave. By means of the Jost function G, characterised by the boundary condition (5.4b), the transmitted wave at $x\to-\infty$ can be expressed in the form

$$\Psi_{\mathrm{OUT}}(x)=G(x\lambda)\begin{Bmatrix} 1/a(\lambda) \\ 0 \end{Bmatrix}\simeq\begin{Bmatrix} [1/a(\lambda)]\exp(-\mathrm{i}\lambda x) \\ 0 \end{Bmatrix}.$$

Introducing the transition matrix T by equations (5.5) and (5.6) in order to relate F and G and using the boundary condition (5.4a), the incoming and reflected waves at $x\to+\infty$ are given by

$$\Psi_{\mathrm{IN}}(x)=F(x\lambda)T(\lambda)\begin{Bmatrix} 1/a(\lambda) \\ 0 \end{Bmatrix}\simeq\begin{Bmatrix} \exp(-\mathrm{i}\lambda x) \\ [b(\lambda)/a(\lambda)]\exp(\mathrm{i}\lambda x) \end{Bmatrix}$$

and we identify the transmission and reflection coefficients

$$t(\lambda)=\frac{1}{a(\lambda)} \qquad r(\lambda)=\frac{b(\lambda)}{a(\lambda)} \qquad |t(\lambda)|^2-|r(\lambda)|^2=1 \tag{5.7}$$

where the last relationship follows from equation (5.6). Notice that, unlike the Schrödinger equation, the eigenvalue problem (5.3) is non-Hermitian and as a consequence the total probability is not conserved, as indicated by equation (5.7). In figure 14 we have shown the scattering solution.

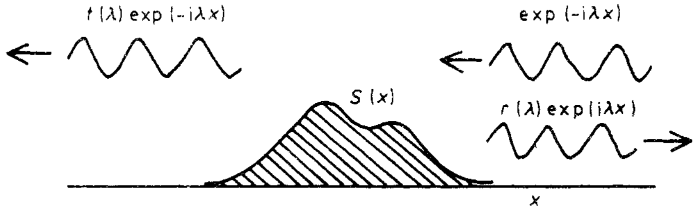

Figure 14. Transmitted and reflected waves for real λ, characterising a scattering solution.

The bound-state solutions correspond to complex values of λ. By means of the Jost function G, whose first column is analytic in the upper-half plane, and the boundary condition (5.4b), we express a bound-state solution with envelope decaying as $\exp(-|x|\mathrm{Im}\lambda)$ for $x \to -\infty$ and $\mathrm{Im}\lambda > 0$, in the form

$$\Psi_{\mathrm{BS}}(x) = G(x\lambda)\begin{Bmatrix}1\\0\end{Bmatrix} \simeq \begin{Bmatrix}\exp(-\mathrm{i}\lambda x)\\0\end{Bmatrix}.$$

For $x \to +\infty$ we obtain, using equations (5.4a), (5.5), and (5.6),

$$\Psi_{\mathrm{BS}}(x) = F(x\lambda)T(\lambda)\begin{Bmatrix}1\\0\end{Bmatrix} \simeq \begin{Bmatrix}a(\lambda)\exp(-\mathrm{i}\lambda x)\\ b(\lambda)\exp(\mathrm{i}\lambda x)\end{Bmatrix}$$

and we must, in order to exclude an increasing solution for $x \to +\infty$, require $a(\lambda) = 0$, i.e. the bound-state spectrum is determined by the zeros of $a(\lambda)$ in the upper-half complex λ plane, or, alternatively, by the poles of the analytically continued transmission coefficient $t(\lambda)$. For 'potentials' $S(x)$ approaching the ground-state value σ^z faster than $|x|^{-2}$ for $|x| \to \infty$ $a(\lambda)$ has only a finite number of zeros λ_n, $n = 1, \ldots, M$, corresponding to a finite number of bound states, in complete analogy with the spectrum of the Schrödinger equation for potentials falling off faster than the Coulomb potential. The normalised bound-state solutions $\Psi_{\mathrm{BS}} \simeq \exp(-\mathrm{i}\lambda_n x)$ for $x \to -\infty$ are furthermore specified by the 'asymptotic characteristics' b_n, i.e. $\Psi_{\mathrm{BS}} \simeq b_n \exp(\mathrm{i}\lambda_n x)$ for $x \to +\infty$ and $n = 1, \ldots, M$. In case λ_n falls within the Bargmann strip we have by analytic continuation $b_n = b(\lambda_n)$ for $|\mathrm{Im}\lambda_n| < \lambda_0$ and $n = 1, \ldots, M$. In figure 15 we have shown the bound-state solution.

The solutions of the eigenvalue equation (5.3) are thus completely characterised by the scattering data

$$\{r(\lambda), -\infty < \lambda < \infty, \mathrm{Im}\,\lambda > 0; \lambda_n, b_n, n = 1, \ldots, M\}. \tag{5.8}$$

Furthermore, by means of the spectral representation

$$a(\lambda) = \exp\left(-\int \frac{\mathrm{d}\mu}{2\pi\mathrm{i}} \frac{\ln(1+|r(\mu)|^2)}{\mu - \lambda - \mathrm{i}\epsilon}\right) \prod_{n=1}^{M} \left(\frac{\lambda - \lambda_n}{\lambda - \lambda_n^*}\right) \tag{5.9}$$

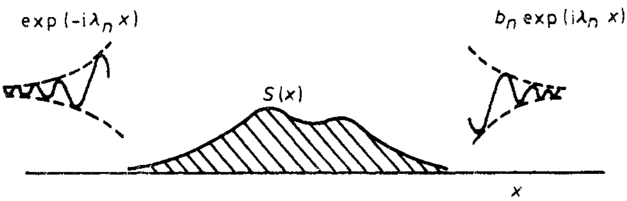

Figure 15. Decaying waves for complex λ, characterising a bound-state solution.

together with equation (5.7) the matrix elements $a(\lambda)$ and $b(\lambda)$, i.e. the transition matrix $T(\lambda)$, are given in terms of the scattering data (5.8). The spectral form (5.9) shows that $a(\lambda)$ has a branch cut along the real axis corresponding to the scattering states, the discontinuity across the cut being determined by the reflection coefficient $r(\lambda)$ and, moreover, allows for an analytic continuation onto the second Riemann sheet where $a(\lambda)$ has poles at $\lambda=\lambda_n^*$, $n=1,\ldots,M$.

5.3. *Time dependence of the scattering data*

In order to determine the time dependence of the scattering data (5.8) or, equivalently, the transition matrix $T(\lambda t)$, induced by the time evolution of the 'potential' $S(xt)$, we consider, using the Jost function G, a solution at $x\to-\infty$ of the form $\Psi(xt)\simeq G(x\lambda)A(t)$. By means of the Lax representation (5.2), the boundary conditions $S\to\sigma^z$ and $\mathrm{d}S/\mathrm{d}x\to 0$ for $|x|\to\infty$, and equation (5.4*b*) we have $A(t)=\exp(2\mathrm{i}\lambda^2 t)A(0)$. For $x\to+\infty$, using equation (5.5), the solution has the form $\Psi(xt)=F(x\lambda)T(\lambda t)A(t)$, and we infers by means of the Lax representation (5.2) and equation (5.4*a*), the time-dependent transition matrix

$$T(\lambda t)=\exp(2\mathrm{i}\lambda^2\sigma^z t)T(\lambda 0)\exp(-2\mathrm{i}\lambda^2\sigma^z t). \tag{5.10}$$

The time evolution of the spin field $S(xt)$ given by the non-linear equation of motion (5.1) thus induces a similarity transformation of $T(\lambda t)$. By means of the identity $\exp(2\mathrm{i}\lambda^2\sigma^z t)=\cos(2\lambda^2 t)+\mathrm{i}\sigma^z\sin(2\lambda^2 t)$ we obtain for the matrix elements $a(\lambda t)$ and $b(\lambda t)$

$$a(\lambda t)=a(\lambda 0)\qquad b(\lambda t)=b(\lambda 0)\exp(-4\mathrm{i}\lambda^2 t) \tag{5.11}$$

or, equivalently, for the scattering data (5.8)

$$r(\lambda t)=r(\lambda 0)\exp(-4\mathrm{i}\lambda^2 t)\qquad -\infty<\lambda<\infty \tag{5.12a}$$

$$\lambda_n(t)=\lambda_n(0)\qquad n=1,\ldots,M \tag{5.12b}$$

$$b_n(t)=b_n(0)\exp(-4\mathrm{i}\lambda_n^2 t)\qquad n=1,\ldots,M \tag{5.12c}$$

in accordance with Takhtajan.

5.4. *The inverse scattering problem*

The reconstruction of the spin field, i.e. the 'potential' $S(xt)$, from the time-dependent scattering data (5.12*a–c*) is called the 'inverse scattering problem' (Faddeev 1963) and is achieved by means of a linear integral equation, the Gel'fand–Levitan–Marchenko equation, presented by Takhtajan and derived in appendix 3. It has the matrix form

$$K(xy;t)+\Phi_1(x+y;t)+\int_x^\infty K(xz;t)\Phi_2(z+y;t)\,\mathrm{d}z=0\qquad \text{for } x\leqslant y \tag{5.13}$$

where the inhomogeneous term Φ_1 and the kernel Φ_2 are given by the time-dependent scattering data, i.e.

$$\Phi_1=\begin{Bmatrix} & -F^* \\ F & \end{Bmatrix}\qquad \Phi_2=-\mathrm{i}\begin{Bmatrix} & \mathrm{d}F^*/\mathrm{d}x \\ \mathrm{d}F/\mathrm{d}x & \end{Bmatrix} \tag{5.14a}$$

$$F(x;t)=\int\frac{r(\lambda t)}{\lambda}\exp(\mathrm{i}\lambda x)\frac{\mathrm{d}\lambda}{2\pi}+\sum_{n=1}^{M}\frac{b_n(t)}{\mathrm{i}a'(\lambda_n)\lambda_n}\exp(\mathrm{i}\lambda_n x). \tag{5.14b}$$

Having in principle solved equation (5.13) for a given set of scattering data, the spin field is subsequently determined by

$$S(xt) = (\mathrm{i}K(xx;t) - \sigma^z)\sigma^z(\mathrm{i}K(xx;t) - \sigma^z)^{-1}. \tag{5.15}$$

The significance of the Gel'fand–Levitan–Marchenko equation (5.13) lies in the fact that it is linear and therefore lends itself to analysis and approximation schemes. In the important case of a 'reflectionless potential', i.e. $r(\lambda) \equiv 0$, equation (5.13) reduces to a set of linear algebraic equations. For $M = 1$, i.e. a single bound state of the Lax operator, Takhtajan obtains the single-soliton solution which, as anticipated, has the same form as the permanent profile solitary wave discussed in § 4. For $M = 2$, corresponding to two bound states of L, he presents the phase and 'centre of mass' shifts induced during two-soliton collisions. In the present context we shall, however, not consider explicit solutions of equation (5.13).

6. Canonical action angle variables

From a general dynamical point of view the canonical representation in § 3 of the continuous Heisenberg chain in configuration space is in a certain sense 'accidental', i.e. we can always envisage a formal canonical mapping $p(x)$, $q(x) \to P, Q$ which, while preserving the Poisson bracket algebra, transforms the model to a dynamically equivalent action angle representation. The transformed Hamiltonian $H'(P)$ then only depends on the canonical momentum P and the equations of motion $\mathrm{d}P/\mathrm{d}t = -\mathrm{d}H'/\mathrm{d}Q$ and $\mathrm{d}Q/\mathrm{d}t = \mathrm{d}H'/\mathrm{d}P$ have the solutions $P = \text{constant}$ and $Q = (\mathrm{d}H'/\mathrm{d}P)t + \text{constant}$ (Landau and Lifshitz 1960).

It is a significant feature of the inverse scattering formalism developed in § 5 that it allows for the explicit construction of an action angle representation. For that purpose the easiest way to proceed is to derive Poisson bracket relations for the scattering data (5.8) or, equivalently, for the transition matrix (5.6). We here follow Zakharov and Manakov (1975) who have applied such techniques to the non-linear Schrödinger and Korteweg–deVries equations. By definition, see equation (2.3),

$$\{T_{ij}(\lambda), T_{mn}(\mu)\} = \int \mathrm{d}x \left[\frac{\mathrm{d}T_{ij}(\lambda)}{\mathrm{d}p(x)} \frac{\mathrm{d}T_{mn}(\mu)}{\mathrm{d}q(x)} - \frac{\mathrm{d}T_{ij}(\lambda)}{\mathrm{d}q(x)} \frac{\mathrm{d}T_{mn}(\mu)}{\mathrm{d}p(x)}\right]$$

or, transforming to the equivalent spin variables S^α and using equation (2.7),

$$\{T_{ij}(\lambda), T_{mn}(\mu)\} = -\sum_{\alpha\beta\gamma} \int \mathrm{d}x \frac{\mathrm{d}T_{ij}(\lambda)}{\mathrm{d}S^\alpha(x)} \frac{\mathrm{d}T_{mn}(\mu)}{\mathrm{d}S^\beta(x)} \epsilon^{\alpha\beta\gamma} S^\gamma(x). \tag{6.1}$$

In appendix 4 we evaluate equation (6.1) and obtain the important intermediate result:

$$\begin{aligned}
\{T_{ij}(\lambda), T_{mn}(\mu)\}& \\
= \tfrac{1}{2}P\frac{\lambda\mu}{\lambda-\mu}&[(\sigma^z T(\lambda))_{ij}(\sigma^z T(\mu))_{mn} - (T(\lambda)\sigma^z)_{ij}(T(\mu)\sigma^z)_{mn}] \\
&+\mathrm{i}\pi\lambda^2\delta(\lambda-\mu)[(\sigma^+ T(\lambda))_{ij}(\sigma^- T(\mu))_{mn} - (\sigma^- T(\lambda))_{ij}(\sigma^+ T(\mu))_{mn} \\
&+(T(\lambda)\sigma^+)_{ij}(T(\mu)\sigma^-)_{mn} - (T(\lambda)\sigma^-)_{ij}(T(\mu)\sigma^+)_{mn}]
\end{aligned} \tag{6.2}$$

(P is the principal value) which shows that the matrix elements $T_{ij}(\lambda)$ satisfy a closed Poisson bracket algebra. All direct reference to configuration space has disappeared and the dynamical behaviour of the continuous Heisenberg chain is now reflected in the Poisson bracket relations (6.2).

For the matrix elements $a(\lambda)$ and $b(\lambda)$ characterising the scattering states, i.e. for real λ, we infer by inspection of equation (6.2) the non-vanishing Poisson brackets

$$\{a(\lambda), b(\mu)\} = -\lambda\mu a(\lambda)b(\mu)\left(P\frac{1}{\lambda-\mu} - \mathrm{i}\pi\delta(\lambda-\mu)\right) \tag{6.3a}$$

$$\{a(\lambda), b(\mu)^*\} = \lambda\mu a(\lambda)b(\mu)^*\left(P\frac{1}{\lambda-\mu} - \mathrm{i}\pi\delta(\lambda-\mu)\right) \tag{6.3b}$$

$$\{b(\lambda), b(\mu)^*\} = 2\lambda^2|a(\lambda)|^2\mathrm{i}\pi\delta(\lambda-\mu). \tag{6.3c}$$

For the scattering data pertaining to the bound-state spectrum, we make use of the implicit equation $a(\lambda_n, \{S\}) = 0$, $n = 1, \ldots, M$, where we have indicated the functional dependence on the 'potential' $S(xt)$. By implicit differentiation with respect to S we have $(\mathrm{d}a/\mathrm{d}\lambda)_{\lambda_n}(\mathrm{d}\lambda_n/\mathrm{d}S) + (\mathrm{d}a/\mathrm{d}S)_{\lambda_n} = 0$, i.e. $\mathrm{d}\lambda_n/\mathrm{d}S = -(\mathrm{d}a/\mathrm{d}\lambda_n)^{-1}_{\lambda_n}(\mathrm{d}a/\mathrm{d}S)_{\lambda_n}$. Using equation (6.2) for λ and μ in the Bargmann strip, see § 5 and appendix 2, and the analytic continuation of equations (6.3*a–c*), we obtain, assuming that the bound-state zeros fall within the Bargmann strip, the non-vanishing Poisson bracket

$$\{\lambda_n, b_n\} = -\left(\frac{\mathrm{d}a}{\mathrm{d}\lambda}\right)^{-1}_{\lambda_n} \lim_{\lambda\to\lambda_n}\{a(\lambda), b_n\} = \lambda_n^2 b_n. \tag{6.4}$$

By means of equations (6.3*a–c*) and (6.4) we are now in a position to construct a new canonical basis for the continuous Heisenberg chain. The variables associated with the scattering states for real λ are given by

$$P(\lambda) = -\frac{1}{\pi\lambda^2}\ln a(\lambda) \qquad P(\lambda) \geqslant 0, -\infty < \lambda < \infty \tag{6.5a}$$

$$Q(\lambda) = -\arg b(\lambda) \qquad -2\pi \leqslant Q(\lambda) \leqslant 0, -\infty < \lambda < \infty \tag{6.5b}$$

and satisfy the canonical Poisson brackets

$$\{P(\lambda), Q(\mu)\} = \delta(\lambda-\mu) \qquad \{P(\lambda), P(\mu)\} = \{Q(\lambda), Q(\mu)\} = 0. \tag{6.6}$$

The canonical variables $P(\lambda)$ and $Q(\lambda)$ are real. The momentum $P(\lambda)$ depends only on $a(\lambda)$ and is therefore, according to equation (5.11), a constant of motion. The constraint $|a(\lambda)|^2 = 1 - |b(\lambda)|^2 \leqslant 1$, given by equation (5.6), furthermore implies that $P(\lambda)$ has a positive range. The coordinate $Q(\lambda)$, defined as the negative phase of $b(\lambda)$, is an angle specified to within a multiple of 2π. From the time dependence of $b(\lambda t)$ given by equation (5.11) we obtain

$$Q(\lambda t) - Q(\lambda 0) = 4\lambda^2 t \qquad -\infty < \lambda < \infty \tag{6.7}$$

i.e. $Q(\lambda t)$ evolves linearly in time with frequency $4\lambda^2$. The canonical variables $P(\lambda)$ and $Q(\lambda t)$ are therefore, in conformity with the general remarks at the beginning of the section, of the action angle type. In a similar manner, the variables pertaining to the bound states for λ in the upper-half complex plane are defined by

$$P_n = \mathrm{i}\lambda_n^{-1} \qquad \mathrm{Re}\, P_n > 0 \qquad n = 1, \ldots, M \tag{6.8a}$$

$$Q_n = \mathrm{i}\ln b_n \qquad n = 1, \ldots, M \tag{6.8b}$$

and obey the canonical Poisson brackets:

$$\{P_n, Q_m\} = \delta_{nm} \qquad \{P_n, P_m\} = \{Q_n, Q_m\} = 0. \tag{6.9}$$

The canonical variables P_n and Q_n are complex. According to equation (5.12*b*) the momentum P_n is a constant of motion. Since, excluding zeros on the real axis, $\mathrm{Im}\,\lambda_n > 0$, the range of $\mathrm{Re}P_n$ is restricted to positive values. The time dependence of the coordinate Q_n is inferred from equation (5.12*a*), i.e.

$$Q_n(t) - Q_n(0) = 4\lambda_n^2 t = -4P_n^{-2} t \qquad n = 1, \ldots, M. \tag{6.10}$$

The canonical variables P_n and $Q_n(t)$ are therefore also of the action angle type.

For later purposes we finally show that the transition matrix $T(\lambda)$ is uniquely determined by the canonical action angle variables. By means of the spectral representation (5.9) and equations (5.7), (6.5*a–b*), and (6.8*a–b*) we obtain

$$a(\lambda) = \exp\left[\mathrm{i} \int \mathrm{d}\mu \frac{\mu^2 P(\mu)}{\mu - \lambda - \mathrm{i}\epsilon}\right] \prod_{n=1}^{M} \left(\frac{\lambda - \mathrm{i}P_n^{-1}}{\lambda + \mathrm{i}(P_n^{-1})^*}\right). \tag{6.11}$$

The inverse scattering method essentially allows for the implicit construction of the non-linear canonical transformation, $S(xt) \rightarrow P(\lambda)$, $Q(\lambda t)$; P_n, $Q_n(t)$, relating the precessional motion in configuration space to the motion of the action angle variables in λ space. Since the derivation of the Poisson bracket relations, in terms of which we have identified the canonical transformation, only involves differential statements, the construction of the action angle representation is a much simpler task than the explicit solution of the Gel'fand–Levitan–Marchenko equation, discussed in § 5.

The dynamical modes of the continuous Heisenberg chain fall in two classes: continuum modes characterised by the canonical variables $P(\lambda)$ and $Q(\lambda t)$ for $-\infty < \lambda < \infty$, and discrete modes specified by the canonical variables P_n and $Q_n(t)$, $n = 1, \ldots, M$. Since $P(\lambda)$ and P_n are constants of motion the continuum modes are characterised by the real distribution $P(\lambda)$, and the discrete modes by the complex numbers P_n, i.e. two real constants of motion for each mode. A given initial spin configuration, say $S(xt=0)$, of the non-linear equation of motion (5.1), 'containing' a distribution $P(\lambda)$ of continuum modes and a set P_n of discrete modes, is furthermore specified by the initial phases $Q(\lambda\, 0)$ and $\mathrm{Re}\, Q_n(0)$ and the initial 'position' $\mathrm{Im}\, Q_n(0)$.

7. The spectrum of solitons and magnons

For the purpose of investigating the physical properties of the continuum and discrete modes associated with the canonical action angle representation, we have in appendix 5 carried out the explicit construction of the Hamiltonian H, the total momentum Π and, corresponding to the boundary condition $S^z \rightarrow 1$ for $|x| \rightarrow \infty$, the z component of the total angular momentum M^z in terms of the action angle variables. We find

$$H = \int \mathrm{d}\lambda\, P(\lambda)\epsilon(\lambda) + \sum_{n=1}^{M} E_n \tag{7.1}$$

$$\Pi = \int \mathrm{d}\lambda\, P(\lambda)\pi(\lambda) + \sum_{n=1}^{M} \Pi_n \tag{7.2}$$

$$M^z = \int \mathrm{d}\lambda\, P(\lambda)m(\lambda) + \sum_{n=1}^{M} M_n \tag{7.3}$$

where

$$\epsilon(\lambda)=4\lambda^2 \qquad \pi(\lambda)=2\lambda \qquad m(\lambda)=-1 \qquad -\infty<\lambda<\infty \tag{7.4}$$

$$E_n=\frac{8\,\mathrm{Re}\,P_n}{|P_n|^2} \qquad \Pi_n=-4\,\mathrm{Im}\ln P_n+2\pi\,\mathrm{sgn}(\mathrm{Im}\,P_n)$$
$$M_n=-2\,\mathrm{Re}\,P_n \qquad n=1,\ldots,M. \tag{7.5}$$

The total energy, momentum and angular momentum are composed of two distinct contributions: a continuum part characterised by the real canonical momentum $P(\lambda)$, $-\infty<\lambda<\infty$, and a discrete part specified by the complex canonical momenta P_n, $n=1,\ldots,M$.

In analogy with the treatment of the sine Gordon equation (Takhtajan and Faddeev 1975) we interpret $P(\lambda)$ as the density of continuum modes on the λ axis. This characterisation is consistent with the action angle nature of the variables $P(\lambda)$ and $Q(\lambda)$ and we conclude that the λth mode in units of $P(\lambda)$ has energy $\epsilon(\lambda)$, momentum $\pi(\lambda)$ and angular momentum $m(\lambda)$, given by equation (7.4). The band of continuum modes can thus be characterised by the quadratic dispersion law

$$\epsilon(\lambda)=\pi(\lambda)^2 \qquad -\infty<\lambda<\infty \tag{7.6}$$

and we identify them tentatively with the magnons or spin waves treated in § 4. Notice, however, that the continuum modes considered here are subject to the fixed boundary condition $S^z\to 1$ for $|x|\to\infty$, unlike the spin waves discussed previously, which have a constant amplitude and are compatible only with periodic boundary conditions. The magnon band is completely characterised by the dispersion law (7.6) and the density $P(\lambda)$ in λ space. With our choice of ground state the angular momentum of the continuum modes is negative and has a magnitude equal to the integrated density $\int P(\lambda)\,\mathrm{d}\lambda$.

The interpretation of the discrete contributions to H, Π and M^z is more straightforward. The nth mode has energy E_n, momentum Π_n and angular momentum M_n, given by equation (7.5). Introducing $P_n=A_n\exp(\mathrm{i}\theta_n)$, where, since $\mathrm{Re}P_n>0$, $|\theta_n|<\pi/2$, we obtain, eliminating A_n and θ_n, the following dispersion law for the nth discrete mode:

$$E_n=\frac{16}{|M_n|}\sin^2\left(\frac{\Pi_n}{4}\right) \qquad |\Pi_n|<2\pi,\ n=1,\ldots,M. \tag{7.7}$$

This expression has exactly the same form as the dispersion law (4.13) for the permanent profile solitary wave discussed in § 4. The discrete modes inferred by the inverse scattering method can thus be identified with the solitary waves and are according to modern terminology called solitons (Scott *et al* 1973). Unlike the magnons or continuum modes which are extended in space, the solitons are spatially localised objects with a width Γ_n given by equation (4.10), $\Gamma_n=8/E_n$, i.e. inversely proportional to their energy. Like the sine Gordon 'breather' modes (Bishop and Schneider 1979), the solitons have internal structure; they carry an angular momentum $-M_n$. In the limit of small momentum $|\Pi_n|\ll 1$, $E_n\simeq\Pi_n^2/|M_n|$, and we can associate an effective mass $|M_n|/2$ with the soliton. The 'rest mass' $|M_n|/2$ is a function of the internal state and is proportional to the angular momentum. For a more detailed discussion of the soliton or solitary wave we refer to § 4 where also the dispersion law (7.7) in figure 12 is plotted for different values of $|M_n|$.

The localised soliton modes and extended magnon modes thus 'diagonalise' the Hamiltonian H and completely exhaust the spectrum of the continuum model (2.6) for $h=0$, i.e. the classical Heisenberg chain in the long-wavelength limit. Since the model in eigenvalue space $\{\lambda\}$ is essentially a gas of non-interacting magnon modes (radiation) and soliton modes (particles) the question of the stability of solitons under collisions in configuration space, investigated numerically by Tjon and Wright (1977), is immediately answered. Under soliton–soliton collision the constants of motion E, Π and M for each individual mode are preserved. Consequently, since the solitons are spatially localised their shape long before and long after a collision is unaltered. In addition to E, Π and M, related by the dispersion law (7.7), the dynamical state of a soliton is characterised by the 'centre of mass' x_0 and the phase ϕ_0 which, not being conserved, do change under collision. As mentioned in § 5, the shifts Δx_0 and $\Delta\phi_0$ in the case of two-soliton collisions have been given in Takhtajan on the basis of the Gel'fand–Levitan–Marchenko equation.

8. The infinite series of conserved densities

Since the Heisenberg chain in the long-wavelength limit is a completely integrable Hamiltonian system, it possesses an infinity of constants of motion, say the canonical momenta P_n and $P(\lambda)$ or, equivalently, the matrix element $a(\lambda)$. What is more interesting, the model has an infinite set of independent constants of motion A_k, $k=1,\ldots,$ and B_k, $k=0,\ldots,$ which have the form of conserved integrated densities, i.e. $A_k=\int \mathrm{d}x\, a_k(x)$ and $B_k=\int \mathrm{d}x\, b_k(x)$, where $a_k(x)$ and $b_k(x)$ are the conserved densities. The recursive procedure for determining the set A_k and B_k is carried out in appendix 6, using the methods developed by Zakharov and Faddeev (1972) for the Korteweg–deVries equation and by Takhtajan and Faddeev (1975) for the sine Gordon equation.

We find

$$\vdots$$

$$a_1(x)=\frac{1}{8}\left[\left(\frac{\mathrm{d}S^x(x)}{\mathrm{d}x}\right)^2+\left(\frac{\mathrm{d}S^y(x)}{\mathrm{d}x}\right)^2+\left(\frac{\mathrm{d}S^z(x)}{\mathrm{d}x}\right)^2\right](=\tfrac{1}{4}\epsilon(x)) \tag{8.1}$$

$$b_0(x)=\frac{1}{4}\,\frac{S^y(x)\,\mathrm{d}S^x(x)/\mathrm{d}x-S^x(x)\,\mathrm{d}S^y(x)/\mathrm{d}x}{1+S^z(x)}(=-\tfrac{1}{2}\pi(x)) \tag{8.2}$$

$$b_1(x)=S^z(x)-1(=m(x)) \tag{8.3}$$

$$b_2(x)=S^x(x)\int_{-\infty}^{x}S^y(y)\,\mathrm{d}y-S^y(x)\int_{-\infty}^{x}S^x(y)\,\mathrm{d}y \tag{8.4}$$

$$b_3(x)=-S^x(x)\int_{-\infty}^{x}\mathrm{d}y\,S^z(y)\int_{-\infty}^{y}\mathrm{d}z\,S^x(z)-S^y(x)\int_{-\infty}^{x}\mathrm{d}y\,S^z(y)\int_{-\infty}^{y}\mathrm{d}z\,S^y(z) \tag{8.5}$$

$$\vdots$$

As expected the series include the energy, momentum and angular momentum densities, i.e. $\epsilon(x)=4a_1(x)$, $\pi(x)=-2b_0(x)$ and $m(x)=b_1(x)$. We notice furthermore that the densities fall into two classes. For $n\geqslant 2b_n(x)$ is a non-local function of the spin density, whereas $b_0(x)$, $b_1(x)$ and $a_n(x)$ for $n\geqslant 1$ are local functions of $S(x)$ and its derivatives. The local densities $a_1\simeq\epsilon$, $b_0\simeq\pi$ and $b_1\simeq m$ are related to the global symmetry transformations: time translation, space translation and spin rotation. The question of whether the other conserved densities are associated with underlying local symmetries of the Heisenberg chain is a fascinating one, but so far essentially unexplored.

9. Summary and conclusion

In this paper we have carried out a detailed analysis of certain aspects of the dynamical behaviour of the classical isotropic Heisenberg chain in the long-wavelength limit. By means of the 'Russian version' of the inverse scattering techniques, we extended the work of Takhtajan (1977) and exhibited, in particular, the canonical action angle representation. In contrast to the sine Gordon equation which has three kinds of modes, namely 'kinks', 'breathers' and 'phonons' (Bishop and Schneider 1979), the spectrum of the Heisenberg chain is exhausted by localised 'magnetic' soliton modes and extended magnon modes.

We remark in passing that the canonical action angle representation allows for a semiclassical quantisation according to standard rules by simply replacing the Poisson brackets by commutators (Landau and Lifshitz 1958, Korenpin and Faddeev 1976). Owing to the uncertainty principle the soliton mode thus becomes delocalised and both solitons and magnons appear as elementary excitations on an equal footing. The quantum Heisenberg chain in the semiclassical limit, i.e. the limit of large S since $S\hbar \to 1$ for $\hbar \to 0$, thus consists of two kinds of non-interacting bosons: spin-one magnons with a quadratic dispersion law $E = p^2$ and solitons with an arbitrary integer spin ν and a dispersion law $E_\nu = 16 \sin^2(P/4)/\nu$, $\nu = 1, \ldots$ ($h = 1$). We notice the interesting feature that the classical soliton band in figure 12 under quantisation breaks up into separate dispersion laws labelled by the spin quantum number ν. Furthermore, in the low-momentum limit $p \ll 1$, i.e. for long wavelengths, the effective soliton mass is quantised. Since, as is well known, the classical canonical transformation does not correspond to a unique quantum-mechanical unitary mapping, the problem of calculating quantum corrections to the semiclassical limit is a subtle one which we shall consider elsewhere.

The action angle representation does also provide the natural starting point for constructing the statistical mechanics of the Heisenberg chain from 'first principles', as well as understanding the influence of perturbations, such as local or exchange anisotropy, impurities, finite lattice distance effects, etc. This point of view has also been stressed by Long and Bishop (1979), who consider the general single solitary wave in the presence of external magnetic and anisotropy fields.

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Appendix 1. The Lax representation

In order to demonstrate the properties of the Lax representation $\mathrm{i}L\Psi = \lambda\Psi$ and $\mathrm{i}\,\mathrm{d}\Psi/\mathrm{d}t = M\Psi$, where $L = S\,\mathrm{d}/\mathrm{d}x$ and $M = 2S\,\mathrm{d}^2/\mathrm{d}x^2 + (\mathrm{d}S/\mathrm{d}x)\,\mathrm{d}/\mathrm{d}x$, we note that the invariance of the spectrum $\{\lambda\}$ implies that the Lax operator L develops according to a similarity transformation $U(t)$, i.e. $L(t)=U(t)L(0)U^{-1}(t)$. Consequently, $\Psi(t)=U(t)\Psi(0)$ and we infer $\mathrm{i}\,\mathrm{d}U/\mathrm{d}t=MU$ or $\mathrm{d}L/\mathrm{d}t=\mathrm{i}[L, M]$. Inserting L and M in the equation of motion for L we obtain

$$\frac{\mathrm{d}S}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}x}=\mathrm{i}S\frac{\mathrm{d}}{\mathrm{d}x}\left[2S\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{\mathrm{d}S}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\right]-\mathrm{i}\left[2S\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{\mathrm{d}S}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\right]S\frac{\mathrm{d}}{\mathrm{d}x}$$

which, using the length condition $S^2=1$, i.e. $S\,\mathrm{d}S/\mathrm{d}x+(\mathrm{d}S/\mathrm{d}x)S=0$ and $S\,\mathrm{d}^2S/\mathrm{d}x^2+(\mathrm{d}^2S/\mathrm{d}x^2)S+2(\mathrm{d}S/\mathrm{d}x)^2=0$, implies the equation of motion (5.1), $\mathrm{d}S/\mathrm{d}t=-\frac{1}{2}\mathrm{i}[S, \mathrm{d}^2S/\mathrm{d}x^2]$, thus proving the above assertion.

Appendix 2. The associated eigenvalue problem

The eigenvalue problem of the Lax operator is given by

$$\mathrm{i}\frac{\mathrm{d}\Psi}{\mathrm{d}x}=\lambda S\Psi \qquad S=\begin{Bmatrix} S^z & S^- \\ S^+ & -S^z \end{Bmatrix} \tag{A2.1}$$

where $S\to\sigma^z$ for $|x|\to\infty$. We note the following general properties of equation (A2.1):

$$\Psi_1 \text{ and } \Psi_2 \text{ solutions then } \Psi_2=\Psi_1 A,\ A \text{ constant} \tag{A2.2a}$$

$$\det\Psi=\text{constant} \tag{A2.2b}$$

$$\Psi \text{ solution for } \lambda \text{ then } \sigma^y\Psi^*\sigma^y \text{ solution for } \lambda^*. \tag{A2.2c}$$

The property (A2.2*a*) is proved by taking a derivative of $\Psi_2^{-1}\Psi_1$ using equation (A2.1). Expressing $\det\Psi$ in the form $\exp(\mathrm{Tr}\ln\Psi)$ and applying $\mathrm{i}\,\mathrm{d}/\mathrm{d}x$ we obtain, using (A2.1) and the cyclic permutability of operators under the trace, $\mathrm{i}\,\mathrm{d}(\det\Psi)/\mathrm{d}x=(\det\Psi)\,\mathrm{Tr}(\lambda S)$ or, since $\mathrm{Tr}\,S=0$, $\det\Psi=\text{constant}$, i.e. the statement (A2.2*b*). Finally, since $S^*=-\sigma^yS\sigma^y$ the property (A2.2*c*) follows by complex conjugation of equation (A2.1).

For $|x|\to\infty$, using $S\to\sigma^z$, the general solution of equation (A2.1) has the form $\exp(-\mathrm{i}\lambda\sigma^z x)A$, A a constant matrix. By means of the method of variations of parameters, i.e. by inserting $\Psi(x)=\exp(-\mathrm{i}\lambda\sigma^z x)A(x)$ in equation (A2.1) and solving for $A(x)$, we obtain the Volterra integral equation

$$\Psi(x)=\exp(-\mathrm{i}\lambda\sigma^z(x-x_0))\Psi(x_0)-\mathrm{i}\lambda\int_{x_0}^{x}\exp(-\mathrm{i}\lambda\sigma^z(x-y))(S(y)-\sigma^z)\Psi(y)\,\mathrm{d}y.$$

For the Jost functions $F(x\lambda)$ and $G(x\lambda)$ defined by the boundary conditions

$$F(x\lambda)\to\exp(-\mathrm{i}\lambda\sigma^z x) \qquad \text{for } x\to+\infty \tag{A2.3a}$$

$$G(x\lambda)\to\exp(-\mathrm{i}\lambda\sigma^z x) \qquad \text{for } x\to-\infty \tag{A2.3b}$$

we have

$$F(x\lambda)=\exp(-\mathrm{i}\lambda\sigma^z x)+\mathrm{i}\lambda\int_{x}^{\infty}\exp(-\mathrm{i}\lambda\sigma^z(x-y))(S(y)-\sigma^z)F(y\lambda)\,\mathrm{d}y \tag{A2.4a}$$

$$G(x\lambda)=\exp(-\mathrm{i}\lambda\sigma^z x)-\mathrm{i}\lambda\int_{-\infty}^{x}\exp(-\mathrm{i}\lambda\sigma^z(x-y))(S(y)-\sigma^z)G(y\lambda)\,\mathrm{d}y. \tag{A2.4b}$$

In order to examine the analytic properties of F and G as functions of λ we perform a Neumann expansion (Courant and Hilbert 1966) of equations (A2.4*a*–*b*):

$$F(x\lambda)=\sum_{n=0}^{\infty} F^{(n)}(x\lambda) \qquad F^{(0)}(x\lambda)=\exp(-\mathrm{i}\lambda\sigma^z x)$$

$$G(x\lambda)=\sum_{n=0}^{\infty} G^{(n)}(x\lambda) \qquad G^{(0)}(x\lambda)=\exp(-\mathrm{i}\lambda\sigma^z x)$$

where

$$F^{(n+1)}(\lambda x)=+\mathrm{i}\lambda\int_x^{\infty}\exp(-\mathrm{i}\lambda\sigma^z(x-y))(S(y)-\sigma^z)F^{(n)}(y\lambda)\,\mathrm{d}y$$

$$G^{(n+1)}(\lambda x)=-\mathrm{i}\lambda\int_{-\infty}^{x}\exp(-\mathrm{i}\lambda\sigma^z(x-y))(S(y)-\sigma^z)G^{(n)}(y\lambda)\,\mathrm{d}y.$$

We next establish the bounds

$$|F^{(n+1)}(x\lambda)|<|\lambda|\int_x^{\infty}\exp(\lambda''\sigma^z(x-y))|S(y)-\sigma^z|F^{(n)}(y\lambda)\,\mathrm{d}y$$

$$|G^{(n+1)}(x\lambda)|<|\lambda|\int_{-\infty}^{x}\exp(\lambda''\sigma^z(x-y))|S(y)-\sigma^z|G^{(n)}(y\lambda)\,\mathrm{d}y$$

where $|A|$ denotes the matrix $|A_{ij}|$ and λ'' is the imaginary part of λ, $\lambda=\lambda'+\mathrm{i}\lambda''$. A first iteration yields

$$|F^{(1)}(x\lambda)|<|\lambda|\exp(\lambda''\sigma^z x)\int_x^{\infty}V(y)\,\mathrm{d}y$$

$$|G^{(1)}(x\lambda)|<|\lambda|\exp(\lambda''\sigma^z x)\int_{-\infty}^{x}V(y)\,\mathrm{d}y$$

where

$$V(y)=\begin{Bmatrix} 1-S^z(y) & \exp(-2\lambda''y)(1-S^z(y)^2)^{1/2} \\ \exp(2\lambda''y)(1-S^z(y)^2)^{1/2} & 1-S^z(y)\end{Bmatrix}.$$

The bounds are controlled by $\int_x^{\infty}V(y)\,\mathrm{d}y$ and $\int_{-\infty}^{x}V(y)\,\mathrm{d}y$ which, provided $|1-S^z(y)|<\exp(-4\lambda_0|y|)$ for $|y|\to\infty$, are convergent in the Bargmann strip $|\lambda''|<\lambda_0$. Moreover, under the weaker condition $|1-S^z(y)|<1/|y|^2$ for $|y|\to\infty$, the second column of $\int_x^{\infty}V(y)\,\mathrm{d}y$ is finite for $\lambda''>0$ and the first column finite for $\lambda''<0$, and vice versa for $\int_{-\infty}^{x}V(y)\,\mathrm{d}y$. Denoting $\int_x^{\infty}V(y)\,\mathrm{d}y=M(x)$, $M'(x)=-V(x)$, we have

$$|F^{(1)}(x\lambda)|<|\lambda|\exp(\lambda''\sigma^z x)M(x)$$

$$|F^{(2)}(x\lambda)|<|\lambda|^2\exp(\lambda''\sigma^z x)\int_x^{\infty}V(y)M(y)\,\mathrm{d}y=|\lambda|^2\exp(\lambda''\sigma^z x)M(x)^2/2$$

and by induction

$$|F^{(n)}(x\lambda)|<|\lambda|^n\exp(\lambda''\sigma^z x)M(x)^n/n!.$$

Similarly

$$|G^{(n)}(x\lambda)|<|\lambda|^n\exp(\lambda''\sigma^z x)N(x)^n/n!$$

where $\int_{-\infty}^{x} V(y)\,\mathrm{d}y = N(x)$. From the bounds for $F^{(n)}$ and $G^{(n)}$ we infer

$$F(x\lambda) = \sum_{n=0}^{\infty} F^{(n)}(x\lambda) < \exp(\lambda''\sigma^z x)\exp(|\lambda| M(x))$$

$$G(x\lambda) = \sum_{n=0}^{\infty} G^{(n)}(x\lambda) < \exp(\lambda''\sigma^z x)\exp(|\lambda| N(x))$$

and it follows that equations (A2.4*a–b*) have solutions and that furthermore, because of uniform convergence, the Jost functions F and G are analytic in the Bargmann strip $|\lambda''| < \lambda_0$. The analyticity domain is larger depending on which matrix elements of F and G we consider. Since

$$M^n(x) = n! \int_x^{\infty} \mathrm{d}x_1\, V(x_1) \int_{x_1}^{\infty} \mathrm{d}x_2\, V(x_2) \ldots \int_{x_{n-1}}^{\infty} \mathrm{d}x_n\, V(x_n)$$

$$N^n(x) = n! \int_{-\infty}^{x} \mathrm{d}x_1\, V(x_1) \int_{-\infty}^{x} \mathrm{d}x_2\, V(x_2) \ldots \int_{-\infty}^{x} \mathrm{d}x_n\, V(x_n)$$

we infer by inspection, inserting $V(x)$, that the columns (M^n_{11}, M^n_{21}) and (N^n_{12}, N^n_{22}) are analytic for $\lambda'' < 0$ and the columns (M^n_{12}, M^n_{22}) and (N^n_{11}, N^n_{21}) analytic for $\lambda'' > 0$, under the weaker condition $|1 - S^z(y)| < 1/|y|^2$ for $|y| \to \infty$. Consequently, (F_{11}, F_{21}) and (G_{12}, G_{22}) are analytic for $\lambda'' < 0$, and (F_{12}, F_{22}) and (G_{11}, G_{21}) analytic for $\lambda'' > 0$.

By inspection of the bounds and by induction we establish that

$$(F_{11}(x\lambda), F_{21}(x\lambda), G_{12}(x\lambda), G_{22}(x\lambda)) < \exp(-|\lambda''|x) \qquad \text{for } \lambda'' < 0$$

$$(F_{12}(x\lambda), F_{22}(x\lambda), G_{11}(x\lambda), G_{21}(x\lambda)) < \exp(-\lambda'' x) \qquad \text{for } \lambda'' > 0.$$

Consequently, by closing the contour for $\lambda'' < 0$:

$$\int \begin{pmatrix} F_{11}(x\lambda) - \exp(-\mathrm{i}\lambda x) \\ F_{21}(x\lambda) \end{pmatrix} \frac{\exp(\mathrm{i}\lambda y)}{\lambda} \frac{\mathrm{d}\lambda}{2\pi} = 0 \qquad \text{for } x > y$$

$$\int \begin{pmatrix} G_{12}(x\lambda) \\ G_{22}(x\lambda) - \exp(\mathrm{i}\lambda x) \end{pmatrix} \frac{\exp(-\mathrm{i}\lambda y)}{\lambda} \frac{\mathrm{d}\lambda}{2\pi} = 0 \qquad \text{for } x < y$$

and, similarly, by closing the contour for $\lambda'' > 0$:

$$\int \begin{pmatrix} F_{12}(x\lambda) \\ F_{22}(x\lambda) - \exp(\mathrm{i}\lambda x) \end{pmatrix} \frac{\exp(-\mathrm{i}\lambda y)}{\lambda} \frac{\mathrm{d}\lambda}{2\pi} = 0 \qquad \text{for } x > y$$

$$\int \begin{pmatrix} G_{11}(x\lambda) - \exp(-\mathrm{i}\lambda x) \\ G_{21}(x\lambda) \end{pmatrix} \frac{\exp(\mathrm{i}\lambda y)}{\lambda} \frac{\mathrm{d}\lambda}{2\pi} = 0 \qquad \text{for } x < y$$

and we infer in matrix form the Jost representations (Takhtajan 1977)

$$F(x\lambda) = \exp(-\mathrm{i}\lambda\sigma^z x) + \lambda \int_x^{\infty} K(xy) \exp(-\mathrm{i}\lambda\sigma^z y)\,\mathrm{d}y \qquad \text{(A2.5}a\text{)}$$

$$G(x\lambda) = \exp(-\mathrm{i}\lambda\sigma^z x) + \lambda \int_{-\infty}^{x} N(xy) \exp(-\mathrm{i}\lambda\sigma^z y)\,\mathrm{d}y. \qquad \text{(A2.5}b\text{)}$$

Here the kernels $K(xy)$ and $N(xy)$ depend functionally on $S(x)$ but are independent of the eigenvalue λ. Since both F and G are solutions of equation (A2.1) we obtain by inserting equations (A2.5*a–b*) partial differential equations for K and N,

$$\frac{\mathrm{d}K(xy)}{\mathrm{d}x}\sigma^z + S(x)\frac{\mathrm{d}K(xy)}{\mathrm{d}y} = 0 \qquad \text{for } x \leqslant y \tag{A2.6a}$$

$$\frac{\mathrm{d}N(xy)}{\mathrm{d}x}\sigma^z + S(x)\frac{\mathrm{d}N(xy)}{\mathrm{d}y} = 0 \qquad \text{for } x \geqslant y \tag{A2.6b}$$

with the boundary conditions

$$S(x) - \sigma^z + \mathrm{i}K(xx) - \mathrm{i}S(x)K(xx)\sigma^z = 0$$

$$S(x) - \sigma^z - \mathrm{i}N(xx) + \mathrm{i}S(x)N(xx)\sigma^z = 0,$$

that is

$$S(x) = (\mathrm{i}K(xx) - \sigma^z)\sigma^z(\mathrm{i}K(xx) - \sigma^z)^{-1} \tag{A2.7a}$$

$$S(x) = (\mathrm{i}N(xx) + \sigma^z)\sigma^z(\mathrm{i}N(xx) + \sigma^z)^{-1}. \tag{A2.7b}$$

The above initial value problems have been considered by Goursat (1964).

Since F and G are solutions of equation (A2.1) the property (A2.2*a*) implies

$$G(x\lambda) = F(x\lambda)T(\lambda) \tag{A2.8}$$

where, using the properties (A2.2*b–c*) and equations (A2.3*a–b*), the transition matrix has the form

$$T(\lambda) = \begin{Bmatrix} a(\lambda) & -(b(\lambda^*))^* \\ b(\lambda) & (a(\lambda^*))^* \end{Bmatrix} \qquad a(\lambda)(a(\lambda^*))^* + b(\lambda)(b(\lambda^*))^* = 1. \tag{A2.9}$$

From the analytic properties of F and G and equations (A2.8) and (A2.9) it follows that $a(\lambda)$ is analytic for $\lambda'' > 0$ provided $S(x) \to \sigma^z$ faster than $1/|x|^2$ for $|x| \to \infty$, and that $a(\lambda)$ and $b(\lambda)$ are analytic in the Bargmann strip $|\lambda''| < \lambda_0$ provided $S(x) \to \sigma^z$ faster than $\exp(-4\lambda_0|x|)$ for $|x| \to \infty$. From equations (A2.5*a–b*), (A2.8) and (A2.9) it follows, furthermore, that $a(\lambda) \to 1$ for $|\lambda| \to \infty$, $\lambda'' > 0$. Consequently, $a(\lambda)$ has only a finite number of zeros λ_n, $n = 1, \ldots, M$, for $\lambda'' > 0$; an infinite number would, since $a(\lambda) \to 1$, imply a limit point, i.e. an essential singularity, in the finite region of the plane $\lambda'' > 0$, thus violating the analycity of $a(\lambda)$.

Consider the function

$$\tilde{a}(\lambda) = a(\lambda) \prod_{n=1}^{M} \left[\frac{\lambda - \lambda_n^*}{\lambda - \lambda_n}\right] \tag{A2.10}$$

which is analytic for $\lambda'' > 0$, has no zeros and approaches unity for $|\lambda| \to \infty$. Since $\tilde{a}(\lambda)$ is an entire function, it follows that $\ln \tilde{a}(\lambda)$ is analytic for $\lambda'' > 0$ and approaches zero for $|\lambda| \to \infty$. For $\lambda'' = 0$, $\ln \tilde{a}(\lambda)$ thus satisfies the Kramers–Kronig relations

$$(\ln \tilde{a}(\lambda))' = P\int \frac{\mathrm{d}\mu}{\pi}\frac{(\ln \tilde{a}(\mu))''}{\mu - \lambda} \qquad (\ln \tilde{a}(\lambda))'' = -P\int \frac{\mathrm{d}\mu}{\pi}\frac{(\ln \tilde{a}(\mu))'}{\mu - \lambda}.$$

It follows from equations (A2.9) and (A2.10), introducing $r(\lambda) = b(\lambda)/a(\lambda)$, that

$$\ln|\tilde{a}(\lambda)| = \ln|a(\lambda)| = -\tfrac{1}{2}\ln(1 + |r(\lambda)|).$$

Consequently, $a(\lambda)$ satisfies the spectral representation (5.9) in § 5, i.e.

$$a(\lambda) = \exp\left(-\int \frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\frac{\ln(1 + |r(\mu)|^2)}{\mu - \lambda - \mathrm{i}\epsilon}\right) \prod_{n=1}^{M} \left(\frac{\lambda - \lambda_n}{\lambda - \lambda_n^*}\right). \tag{A2.11}$$

Appendix 3. The Gel'fand–Levitan–Marchenko equation

In order to derive the Gel'fand–Levitan–Marchenko equation we use equations (A2.8) and (A2.9). Introducing $r(\lambda)=b(\lambda)/a(\lambda)$,

$$(a(\lambda)^{-1}-1)G_{11}(x\lambda)=F_{11}(x\lambda)+F_{12}(x\lambda)r(\lambda)-G_{11}(x\lambda)$$
$$(a(\lambda)^{-1}-1)G_{21}(x\lambda)=F_{21}(x\lambda)+F_{22}(x\lambda)r(\lambda)-G_{21}(x\lambda).$$

Inserting equations (A2.5*a–b*), multiplying by $\exp(\mathrm{i}\lambda z)/\lambda$, and integrating over λ, we obtain

$$\int \exp(\mathrm{i}\lambda z)G_{11}(x\lambda)(a(\lambda)^{-1}-1)\lambda^{-1}\frac{\mathrm{d}\lambda}{2\pi}$$
$$=\int \exp(\mathrm{i}\lambda z)\left[\int_x^\infty K_{11}(xy)\exp(-\mathrm{i}\lambda y)\,\mathrm{d}y+r(\lambda)\int_x^\infty K_{12}(xy)\exp(\mathrm{i}\lambda y)\,\mathrm{d}y\right.$$
$$\left.-\int_{-\infty}^x N_{11}(xy)\exp(-\mathrm{i}\lambda y)\,\mathrm{d}y\right]\frac{\mathrm{d}\lambda}{2\pi}$$

$$\int \exp(\mathrm{i}\lambda z)G_{21}(x\lambda)(a(\lambda)^{-1}-1)\lambda^{-1}\frac{\mathrm{d}\lambda}{2\pi}$$
$$=\int \exp(\mathrm{i}\lambda z)\left[\lambda^{-1}r(\lambda)\exp(\mathrm{i}\lambda x)+r(\lambda)\int_x^\infty K_{22}(xy)\exp(\mathrm{i}\lambda y)\,\mathrm{d}y\right.$$
$$\left.+\int_x^\infty K_{21}(xy)\exp(-\mathrm{i}\lambda y)\,\mathrm{d}y-\int_{-\infty}^x N_{21}(xy)\exp(-\mathrm{i}\lambda y)\,\mathrm{d}y\right]\frac{\mathrm{d}\lambda}{2\pi}.$$

Since G_{11} and G_{21} are analytic for $\lambda''>0$ and fall off faster than $\exp(-\lambda''x)$ and $a(\lambda)\to 1$ for $|\lambda|\to\infty$ we close for $z>-x$ the contour for $\lambda''>0$. Using $\int\exp(\mathrm{i}\lambda x)\,\mathrm{d}\lambda/2\pi=\delta(x)$ we have

$$\mathrm{i}\sum_{n=1}^{M}\frac{\exp(\mathrm{i}\lambda_n z)G_{11}(x\lambda_n)}{\lambda_n a'(\lambda_n)}=K_{11}(xz)-N_{11}(xz)$$
$$+\int_x^\infty K_{12}(xy)\left(\int\exp(\mathrm{i}\lambda(y+z))r(\lambda)\frac{\mathrm{d}\lambda}{2\pi}\right)\mathrm{d}y$$

$$\mathrm{i}\sum_{n=1}^{M}\frac{\exp(\mathrm{i}\lambda_n z)G_{21}(x\lambda_n)}{\lambda_n a'(\lambda_n)}$$
$$=K_{21}(xz)-N_{21}(xz)+\int_x^\infty K_{22}(xy)\left(\int\exp(\mathrm{i}\lambda(y+z))r(\lambda)\frac{\mathrm{d}\lambda}{2\pi}\right)\mathrm{d}y$$
$$+\int\frac{\mathrm{d}\lambda}{2\pi}\lambda^{-1}r(\lambda)\exp(\mathrm{i}\lambda(x+z))$$

where $a'(\lambda_n)=[\mathrm{d}a(\lambda)/\mathrm{d}\lambda]_{\lambda=\lambda_n}$. At the zeros of $a(\lambda)$ we obtain, using equations (A2.5*a–b*), (A2.8) and (A2.9), and assuming that λ_n falls within the Bargmann strip $|\lambda''|<\lambda_0$, i.e. $b_n=b(\lambda_n)$,

$$G_{11}(x\lambda_n)=b_n\lambda_n\int_x^\infty K_{12}(xy)\exp(\mathrm{i}\lambda_n y)\,\mathrm{d}y$$

$$G_{21}(x\lambda_n)=b_n\exp(\mathrm{i}\lambda_n x)+b_n\lambda_n\int_x^{\infty}K_{22}(xy)\exp(\mathrm{i}\lambda_n y)\,\mathrm{d}y$$

which, by insertion for $z>x$, yield the Gel'fand–Levitan–Marchenko equations

$$K_{11}(xz)+\int_x^{\infty}K_{12}(xy)\Big(\int\exp(\mathrm{i}\lambda(y+z))r(\lambda)\frac{\mathrm{d}\lambda}{2\pi}+\sum_{n=1}^{M}\exp(\mathrm{i}\lambda_n(y+z))(-\mathrm{i})\frac{b_n}{a'(\lambda_n)}\Big)\,\mathrm{d}y=0 \tag{A3.1a}$$

$$\begin{aligned}K_{21}(xz)&+\Big(\int\exp(\mathrm{i}\lambda(x+z))\frac{r(\lambda)}{\lambda}\frac{\mathrm{d}\lambda}{2\pi}+\sum_{n=1}^{M}\exp(\mathrm{i}\lambda_n(x+z))(-\mathrm{i})\frac{b_n}{a'(\lambda_n)\lambda_n}\Big)\\&+\int_x^{\infty}K_{22}(xy)\Big(\int\exp(\mathrm{i}\lambda(y+z))r(\lambda)\frac{\mathrm{d}\lambda}{2\pi}\\&+\sum_{n=1}^{M}\exp(\mathrm{i}\lambda_n(y+z))(-\mathrm{i})\frac{b_n}{a'(\lambda_n)}\Big)\,\mathrm{d}y=0.\end{aligned} \tag{A3.1b}$$

Similarly, another set of integral equations can be derived for K_{12} and K_{22}. Compactly, the Gel'fand–Levitan–Marchenko equation for the Jost kernel K can be expressed in the form given by equations (5.13) and (5.14*a*–*b*) in § 5, in agreement with Takhtajan (1977). Given the solution of the Gel'fand–Levitan–Marchenko equation for a set of scattering data $\{r(\lambda);\ \lambda_n,\ b_n\}$ the spin density $S(x)$ is determined by equation (A2.7*a*).

Appendix 4. The action angle representation

In order to evaluate the Poisson bracket

$$\{T_{ij}(\lambda),\ T_{mn}(\mu)\}=-\sum_{\alpha\beta\gamma}\int\mathrm{d}x\frac{\mathrm{d}T_{ij}(\lambda)}{\mathrm{d}S^{\alpha}(x)}\frac{\mathrm{d}T_{mn}(\mu)}{\mathrm{d}S^{\beta}(x)}\epsilon^{\alpha\beta\gamma}S^{\gamma}(x) \tag{A4.1}$$

we use equation (A2 8), the identity $\delta F^{-1}F+F^{-1}\delta F=0$, and from equations (A2.5*a*–*b*) the property $\mathrm{d}F(y\lambda)/\mathrm{d}S^{\alpha}(x)=\mathrm{d}G(x\lambda)/\mathrm{d}S^{\alpha}(y)=0$ for $y>x$, i.e.

$$\frac{\mathrm{d}T(\lambda)}{\mathrm{d}S^{\alpha}(x)}=F^{-1}(y\lambda)\frac{\mathrm{d}G(y\lambda)}{\mathrm{d}S^{\alpha}(x)}\qquad\text{for } y>x. \tag{A4.2}$$

By differentiation of equation (A2.1) $\mathrm{d}G/\mathrm{d}S^{\alpha}$ satisfies the Green's function equation

$$\Big(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}y}-\lambda S(y)\Big)\frac{\mathrm{d}G(y\lambda)}{\mathrm{d}S^{\alpha}(x)}=\lambda\sigma^{\alpha}G(y\lambda)\delta(x-y). \tag{A4.3}$$

Integrating over the interval $x-\epsilon$ to $x+\epsilon$ and letting $\epsilon\to 0$, we obtain the boundary condition $\mathrm{i}\mathrm{d}G(x^{+}\lambda)/\mathrm{d}S^{\alpha}(x)=\lambda\sigma^{\alpha}G(x\lambda)$ and in terms of G the solution of equation (A4.3),

$$\frac{\mathrm{d}G(y\lambda)}{\mathrm{d}S^{\alpha}(x)}=-\mathrm{i}\lambda G(y\lambda)G^{-1}(x\lambda)\sigma^{\alpha}G(x\lambda)\qquad\text{for } y>x \tag{A4.4}$$

which inserted in equation (A4.2) for $y=x^{+}$ yields

$$\frac{\mathrm{d}T(\lambda)}{\mathrm{d}S^{\alpha}(x)}=-\mathrm{i}\lambda F^{-1}(x\lambda)\sigma^{\alpha}G(x\lambda). \tag{A4.5}$$

Substituting equation (A4.5) in equation (A4.1) and using the identity

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{\alpha}(F^{-1}(x\lambda)\sigma^{\alpha}G(x\lambda))_{ij}(F^{-1}(x\mu)\sigma^{\alpha}G(x\mu))_{mn}\right]$$

$$=2(\lambda-\mu)\sum_{\alpha\beta\gamma}\epsilon^{\alpha\beta\gamma}(F^{-1}(x\lambda)\sigma^{\alpha}G(x\lambda))_{ij}(F^{-1}(x\mu)\sigma^{\beta}G(x\mu))_{mn}S^{\gamma}(x)$$

which follows from equation (A2.1), we obtain, performing the integration and controlling the limits of integration by means of the cutoff L,

$$(\lambda-\mu)\{T_{ij}(\lambda),T_{mn}(\mu)\}$$
$$=\tfrac{1}{2}\lambda\mu\lim_{L\to\infty}\sum_{\alpha}[(F^{-1}(L\lambda)\sigma^{\alpha}G(L\lambda))_{ij}(F^{-1}(L\mu)\sigma^{\alpha}G(L\mu))_{mn}-(L\to -L)].$$

Finally, using equations (A2.3*a*–*b*) and (A2.8) and for $\lambda''=0$, $\lim_{L\to\infty}P\exp(\mathrm{i}\lambda L)=\pi\delta(x)$, introducing $\sigma^{\pm}=\sigma^{x}\pm\mathrm{i}\sigma^{y}$ and performing a bit of Pauli matrix algebra we obtain equation (6.2) in § 6.

Appendix 5. Energy-momentum-angular momentum

The form of the real Hamiltonian (7.1) is inferred from equations (6.7) and (6.10) in conjunction with the equation of motion $\mathrm{d}Q/\mathrm{d}t=\{H,Q\}$ and $\mathrm{d}P/\mathrm{d}t=\{H,P\}$ and equations (6.6) and (6.9), i.e.

$$H=\int \mathrm{d}\lambda P(\lambda)(2\lambda)^2+\sum_{n=1}^{M}4(P_n^{-1}+(P_n^{-1})^*). \qquad (A5.1)$$

The form of the real total momentum (7.2) is obtained using equation (3.5), i.e.

$$\{T(\lambda),\Pi\}=\int \mathrm{d}x\sum_{\alpha}\frac{\mathrm{d}T(\lambda)}{\mathrm{d}S^{\alpha}(x)}\{S^{\alpha}(x),\Pi\}=\int \mathrm{d}x\sum_{\alpha}\frac{\mathrm{d}T(\lambda)}{\mathrm{d}S^{\alpha}(x)}\frac{\mathrm{d}S^{\alpha}(x)}{\mathrm{d}x}. \qquad (A5.2)$$

Inserting equation (A4.5), performing a partial integration using equation (A2.1), $S^2=I$, and $S\to\sigma^z$ for $|x|\to\infty$, and by the aid of equations (A2.8) and (A2.3*a*–*b*)

$$\{T(\lambda),\Pi\}=-\mathrm{i}\lambda[\sigma^zT(\lambda)-T(\lambda)\sigma^z] \qquad \text{for } \lambda''=0. \qquad (A5.3)$$

By analytic continuation into the Bargmann strip using equation (A2.9), $\{b(\lambda),\Pi\}=2\mathrm{i}\lambda b(\lambda)$ and $\{b_n,\Pi\}=2\mathrm{i}\lambda_nb_n$ or, by equations (6.5*b*) and (6.8*a*–*b*),

$$\{Q(\lambda),\Pi\}=-2\lambda,\ -\infty<\lambda<\infty \qquad \{Q_n,\Pi\}=-2\mathrm{i}P_n^{-1},\ n=1,\ldots,M. \qquad (A5.4)$$

Finally, using equations (6.6) and (6.9) the total momentum must have the form

$$\Pi=\int \mathrm{d}\lambda\, P(\lambda)2\lambda+\sum_{n=1}^{M}2\mathrm{i}[\ln P_n-\ln P_n^*-\mathrm{i}\pi\,\mathrm{sgn}(\mathrm{Im}\,P_n)] \qquad (A5.5)$$

where the branch of the logarithm is chosen such that the discrete contributions to H and Π vanish for $\mathrm{Re}P_n\to 0$.

The real total angular momentum (7.3) is determined using equations (2.7) and (3.7), i.e.

$$\{T(\lambda),M^{\alpha}\}=\int \mathrm{d}x\{T(\lambda),S^{\alpha}(x)\}=\sum_{\beta\gamma}\epsilon^{\alpha\beta\gamma}\int \mathrm{d}x\frac{\mathrm{d}T(\lambda)}{\mathrm{d}S^{\beta}(x)}S^{\gamma}(x). \qquad (A5.6)$$

Inserting equation (A4.5), using the identity

$$\frac{\mathrm{d}}{\mathrm{d}x}(F^{-1}(x\lambda)\sigma^{\alpha}G(x\lambda)) = -2\lambda \sum_{\beta\gamma} \epsilon^{\alpha\beta\gamma} F^{-1}(x\lambda)\sigma^{\beta}G(x\lambda)S^{\gamma}(x)$$

which follows from equation (A2.1), and integrating, we have for $\lambda''=0$

$$\{T(\lambda), M^{\alpha}\} = \frac{\mathrm{i}}{2}[F^{-1}(\infty\lambda)\sigma^{\alpha}G(\infty\lambda) - F^{-1}(-\infty\lambda)\sigma^{\alpha}G(-\infty\lambda)].$$

Since $S \to \sigma^z$ for $|x| \to \infty$ only $\{T(\lambda), M^z\}$ is well defined. Using equations (A2.8) and (A2.3*a*–*b*)

$$\{T(\lambda), M^z\} = \frac{\mathrm{i}}{2}[\sigma^z T(\lambda) - T(\lambda)\sigma^z]. \tag{A5.7}$$

In the Bargmann strip by the aid of equation (A2.9) $\{b(\lambda), M^z\} = -\mathrm{i}b(\lambda)$ and $\{b_n, M^z\} = -\mathrm{i}b_n$, i.e. by equations (6.5*b*) and (6.8*b*):

$$\{Q(\lambda), M^z\} = 1, -\infty < \lambda < \infty \qquad \{Q_n, M^z\} = 1, n = 1, \ldots, M \tag{A5.8}$$

and we infer, using equations (6.6) and (6.9),

$$-M^z = \int \mathrm{d}\lambda\, P(\lambda) + \sum_{n=1}^{M} (P_n + P_n^*). \tag{A5.9}$$

Appendix 6. The constants of motion

In order to determine the infinite set of conserved integrated densities, we expand the real constant of motion $\operatorname{Im} \ln a(\lambda)$ in powers of λ and $1/\lambda$ for $\lambda''=0$. By means of equation (6.11*a*) we obtain

$$\operatorname{Im}(\ln a(\lambda)) = -\sum_{k=1}^{\infty} \lambda^{-k} A_k \qquad \text{for } |\lambda| \to \infty \tag{A6.1a}$$

$$\operatorname{Im}(\ln a(\lambda)) = -\sum_{k=0}^{\infty} \lambda^{k} B_k \qquad \text{for } |\lambda| \to 0 \tag{A6.1b}$$

where

$$A_k = \int \mathrm{d}\mu\, P(\mu)\mu^{1+k} + \mathrm{i}\frac{(-1)^{1+k}}{k} \sum_{n=1}^{M} \mathrm{i}^k[(-1)^k P_n^{-k} - (P_n^*)^{-k}] \tag{A6.2a}$$

$$B_0 = -\int \mathrm{d}\mu\, P(\mu)\mu + \mathrm{i} \sum_{n=1}^{M} [\ln P_n^* - \ln P_n + \mathrm{i}\pi \operatorname{sgn}(\operatorname{Im} P_n)] \tag{A6.2b}$$

$$B_k = -\int \mathrm{d}\mu\, P(\mu)\mu^{1-k} + \mathrm{i}\frac{(-1)^{1+k}}{k} \sum_{n=1}^{M} \mathrm{i}^{-k}[(-1)^k P_n^k - (P_n^*)^k]. \tag{A6.2c}$$

In configuration space the independent constants of motion A_k and B_k are derived by a recursive procedure. Introducing

$$\sigma(x\lambda) = \frac{\mathrm{d}\ln G_{11}(x\lambda)}{\mathrm{d}x} + \mathrm{i}\lambda \tag{A.6.3}$$

we have, using equations (A2.8), (A2.9) and (A2.3*a–b*),

$$\operatorname{Im}\ln a(\lambda)=\int \operatorname{Im}\sigma(x\lambda)\,\mathrm{d}x. \tag{A6.4}$$

The density $\sigma(x\lambda)$ is determined by considering equation (A2.1) for G_{11} and introducing $\Phi(x\lambda)=G_{21}(x\lambda)/G_{11}(x\lambda)$, i.e. by equation (A6.3)

$$\sigma(x\lambda)=-\mathrm{i}\lambda\,(S^z(x)-1+S^-(x)\Phi(x\lambda)) \tag{A6.5}$$

where $\Phi(x\lambda)$ satisfies the non-linear generalised Riccati equation (Ince 1956)

$$\mathrm{i}\frac{\mathrm{d}\Phi(x\lambda)}{\mathrm{d}x}+\lambda[S^-(x)\Phi(x\lambda)^2+2S^z(x)\Phi(x\lambda)-S^+(x)]=0. \tag{A6.6}$$

Inserting $\Phi(x\lambda)=\Sigma_{n=1}^{\infty}\lambda^n f_n(x)$ and $\Phi(x\lambda)=\Sigma_{n=0}^{\infty}\lambda^{-n}g_n(x)$ in equation (A6.6) and using $\Phi(-\infty,\lambda)=0$, we obtain the recursion formulae

$$\mathrm{i}\frac{\mathrm{d}f_n}{\mathrm{d}x}+S^-\sum_{p=0}^{n-1}f_pf_{n-p-1}+2S^zf_{n-1}-S^+\delta_{n1}=0 \qquad f_0=0 \tag{A6.7a}$$

$$\mathrm{i}\frac{\mathrm{d}g_n}{\mathrm{d}x}+S^-\sum_{p=0}^{n+1}g_pg_{n-p+1}+2S^zg_{n+1}=0 \qquad g_0=(1-S^z)/S^-. \tag{A6.7b}$$

By straightforward iteration we find

$$\Phi(x\lambda)=-\mathrm{i}\lambda\int_{-\infty}^{x}S^+(y)\,\mathrm{d}y-\lambda^2\int_{-\infty}^{x}\mathrm{d}y\,S^z(y)\int_{-\infty}^{y}\mathrm{d}z\,S^+(z)+\dots \qquad \text{for } |\lambda|\to 0$$

$$\Phi(x\lambda)=\frac{1-S^z(x)}{S^-(x)}-\frac{\mathrm{i}}{2\lambda}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1-S^z(x)}{S^-(x)}\right)+\dots \qquad \text{for } |\lambda|\to\infty$$

which, by substitution in equations (A6.5) and (A6.4), yield the conserved densities $a_n(x)$ and $b_n(x)$ associated with the constants of motion A_n and B_n, see § 8.

Note added in proof. After the completion of the present paper the author became aware of a paper by Zakharov and Takhtajan (1979) on the equivalence of the non-linear Schrödinger equation and the equation of the Heisenberg ferromagnet, which contains some of the material presented here, in particular the recursive procedure for the determination of the series of constants of motion.

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